FEATURE

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Flavors of Geometry Motivated by Mathematics and Physics

Geometry is one of the oldest branches of mathematics. Its goal is to describe the structure of space, so it is motivated by our physical understanding of the space around us. From the human perspective, geometry is dominated by the visual sense, the source of so-called geometric intuition, which allows us to see the connections between parts of an image before formally describing them. It is our everyday experience of seeing everyday things around us that forms the basis of our assumptions about what geometry should be about.

But modern physics tells us that the structure of our physical space (or space-time) at very short distances in not known. Why? Because our visual sense is based on the eye observing the light reflected by objects. But light is in fact a wave phenomenon. So "objects" smaller than the wave length of the light, are hard to observe precisely in this way. Trying to increase the the resolution, so to say, involves bombarding the object with quanta of light (or with other particles) with higher and higher energies. Since our ability to produce higher energy particles is limited, so is our ability to "see" at very short distances.

So we cannot just assume that at extremely small ranges our space has the same general structure as we are used to from everyday life. For example, it is not a law of nature that it consists of points which are arranged continuously next to each other, with well defined distances between them, etc. In other words, the premises on which we base our development of geometry must be re-examined.

There have traditionally been several branches of geometry: differential geometry, algebraic geometry, topology, combinatorial geometry and so on. But all these types agreed on certain assumptions (like the above) on what should we understand by a "space" in the first place, and differed only in the approaches they took in studying it: differential calculus, algebraic equations and so on.

However, in recent decades, there appeared several new directions of geometry which require a change in the very way we think about geometric shapes. I want to discuss some of these directions.

Grothendieck's scheme theory

Most of new geometric approaches are based on the theory of schemes developed by A. Grothendieck in the early 1960's as a new foundation of algebraic geometry. The basic idea is that all the information about a "space" X (whatever we mean by this) must be encoded by the datum R of functions on X. In the naive settings R consists of functions on X of certain kind, i.e., of rules f associating to any point x of X some numerical value $f(x)$. Such functions can be added, subtracted and multiplied pointwise. Mathematically, a system of functions (or other entities) closed under addition, subtraction and

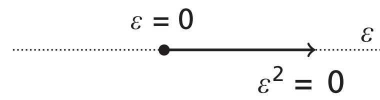


Figure 1: A non-classical space given by the equation $\varepsilon^2 = 0$.

multiplication is called a *ring*. The rings of functions have one obvious but important property: the multiplication is commutative:

$$f \cdot g = g \cdot f.$$

This property turns out to be crucial for the formalism of algebraic geometry to work.

For example, if X is the n-dimensional coordinate space, then the corresponding R consists of polynomials in n variables, expressions like

$$f(x, y) = 2x + 3y + 16x^5y^3 + x^4y^7$$

(here $n = 2$). If X is given by a polynomial equation, then the corresponding R is obtained from the polynomial ring by a natural identification: we "identify" (consider as one) any two polynomials f, g whose difference is a multiple of the equation, and similarly for the case of several equations. One refers to X as the *spectrum* of R and writes $X = \mathrm{Spec}(R)$.

The word "spectrum" comes from spectral theory of linear operators (i.e., the theory of eigenvalues), an area that was heavily influenced by the needs of quantum physics. So this is an example of implicit influence of physics on mathematics.

The important (and initially controversial) step in Grothendieck's theory is that one can associate the geometric image (scheme) Spec (R) to any commutative ring R whatsoever. An important non-classical example is given by the *ring of dual numbers* D. An element of this ring is an expression $a + b\varepsilon$; such expressions are multiplied formally using the rule $\varepsilon^2 = 0$. So $\varepsilon^2 = 0$ is the equation in this case. How is it different from $\varepsilon = 0$? Naively, D cannot be realized as the set of functions on anything, because there is no number other than 0 which squares to 0. Nevertheless, the scheme Spec (D) has a meaningful geometric interpretation: it is viewed as having one point (where $\varepsilon = 0$) and also having the tangent direction at this point, but no further data, see Fig. 1. In a way, this is a revival of the old idea of "infinitesimally small quantities": ε itself is not yet zero, but is "so small" that ε^2 is already negligible. One can also consider infinitesimals ε such that $\varepsilon^2 \neq 0$ but some higher power of ε vanishes. Rings containing such infinitesimals (called *nilpotents*) are visualized as corresponding to infinitesimally thin neighborhoods of more classical geometric images (curves et cetera).

Noncommutative geometry

The spectacular success of "visualization of commutative rings" given by scheme theory led to repeated attempts to extend it to *noncommutative rings*, algebraic structures in which the multiplication can lead to $f \cdot g \neq g \cdot f$.

A non-mathematician may wonder: what is the importance of such structures? do they really appear "in real life"? In fact, it was the advent of quantum mechanics which brought noncommutative rings into the forefront of physics. Usual physical quantities are promoted, in quantum mechanics, to non-commuting "operators". A typical commutation relation is $p \cdot q - q \cdot p = i\hbar$ between the operators

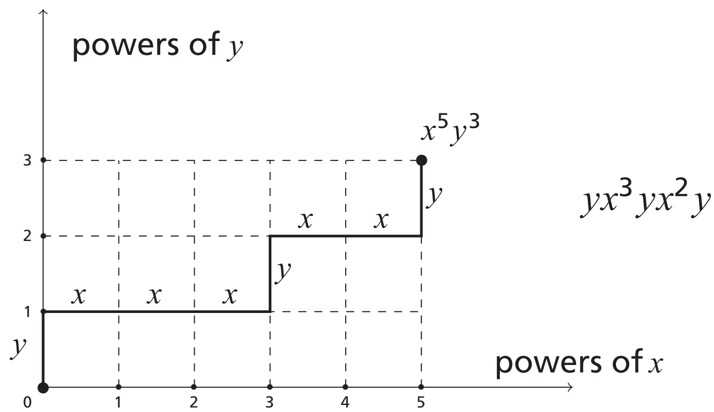


Figure 2: A noncommutative monomial represented by a path.

corresponding to the coordinate and momentum of a particle. There are many other examples in pure mathematics, such as multiplication of matrices, or composing transformations, i.e., operations of some kind. In fact, if we do any actions in a sequence, the result usually depends on the order. To put on a shirt and then a jacket is not the same as to first put on a jacket and then a shirt!

A good illustration of the challenges presented by noncommutativity is provided by the concept of "noncommutative polynomials". For instance, consider two variables x, y which do not commute. Then we have 4 quadratic monomials: x^2, xy, yx, y^2, all different. If we think of x, y as commuting, then xy and yx are the same but as noncommutative monomials they are different. In this way, a single commutative monomial can be represented by several noncommutative ones. For example x^5y^3 can be lifted to yx^3yx^2y, or to xyx^2yxyx, or to several others. It is convenient to make a picture (known as *Newton's diagram*) depicting a usual monomial, say x^5y^3 by a point on the plane with coordinates (5, 3). Then a noncommutative lifting of this monomial corresponds to a "taxicab path" (like in a city with a grid of street blocks) starting from (0, 0) and ending at (5, 3). That is, one move to the east corresponds to x and one move to the north corresponds to y, see Fig. 2.

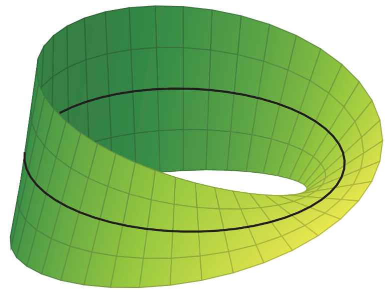
Thus a noncommutative polynomial is really a sum over such paths. Making x and y commute means that we perform *summation over paths* with fixed beginning and end. This means that "commutativization" (forcing noncommutative rings to be commutative) can be seen as an algebraic analog of path integration which is the fundamental conceptual tool of modern physics.

One way of attaching geometric intuition to noncommutative rings is to use the concept of a *vector bundle* in geometry, i.e., a continuous family of vector spaces parametrized by a space X. For example, the Moebius strip (Fig.3) is a vector bundle over its central (black) circle: a family of "vertical" lines parametrized by it. If X corresponds to a ring R, then a vector bundle on X gives rise to an algebraic object M called a *module* over R, where we can multiply elements r of R and m of M and get an element $m' = r \cdot m$ of M.

Attaching geometric intuition to noncommutative rings is not only a tool for studying such rings, it has many applications to more familiar geometric problems. In many cases, one can approximate a usual ("commutative") but complicated or badly behaved space, by a much simpler non-commutative object.

Super-geometry

Still, noncommutative algebraic structures do not



$r \cdot m = m'$

Figure 3: The Moebius strip[1] and the multiplication in a module.

*1 Picture source: pgfplots.

seem to be capable of fully geometric interpretation. It is the commutativity property that makes many essential constructions work.

An alternative approach is to look for properties "similar to commutativity" which act as similar but different keys to the realm of geometry. One of such properties is graded, or super-commutativity, which leads to super-geometry.

In this setting we have a ring R having quantities of two types, even and odd. A general element of R is represented as a sum of an odd and an even one. The super-commutativity law (also known as the Koszul sign law) reads:

$$f \cdot g = (-1)^{\deg(f) \cdot \deg(g)} g \cdot f, \quad (1)$$

where f and g are either even or odd. The quantity $\deg(f)$ is equal to 0 for f even and 1 for f odd. In other words, we have $f \cdot g = g \cdot f$ when at least one of f, g is even and $f \cdot g = -g \cdot f$ when both are odd.

So a super-commutative ring is not commutative in the usual sense. Nevertheless, the experience of mathematicians has been that super-commutativity unlocks all the geometric features that can be associated to usual commutative rings. For example, one can speak about super-manifolds, objects which have usual (even, commuting) coordinates together with odd, anti-commuting coordinates.

The super-commutative law is just one of an elaborate system of sign rules in this kind of algebra (sometimes called super-algebra). The remarkable fact is that these rules are non-contradictory: various transformations incur various sign changes but it never happens that doing something in two different ways results in different signs (that would destroy the whole theory). This almost mystical self-consistency of the rules adds a lot to the appeal of the theory. But perhaps the real reason the things do not collapse is the physical origins of super-algebra.

It is known in physics that elementary particles fall into two types: bosons and fermions. The difference is that more than one fermion cannot be in the same quantum state (this is known as the Pauli exclusion principle), while for bosons it is possible. For example, electron and proton are fermions, while photon (the quantum of light) is a boson. At a more mathematical level, the state vector of a system of several fermions changes sign under permuting any two of them, in a way remindful of (1). This fermionic nature of electrons is at the basis of the structure of atoms and chemical elements, and so is fundamental for the existence of the universe as we know it.

The idea of super-geometry was first suggested by F. Berezin in the late 1960s. He had a clear physical motivation: to create geometry that would account for the behavior of fermionic particles. This was included into a large program of so-called *supersymmetry*, i.e., symmetry between bosonic and

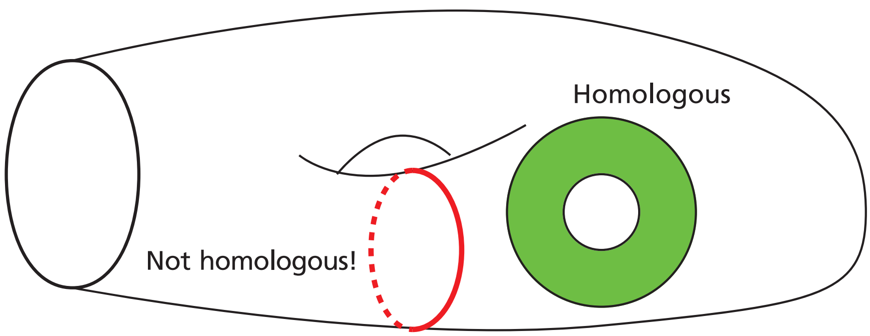


Figure 4: Homologous and non-homologous cycles.

fermionic particles. Most importantly, these ideas turned out to be extremely fruitful in string theory. The ± signs that appear in super-algebra, far from being an annoyance, ended up serving a crucial purpose: they make answers provided by physical theories more manageable (free from obvious infinities). That is, individual terms in the calculations end up being very large but because of summation with ± signs the end results appear finite, something that would not happen without super-algebra.

Modern super-geometry serves as a geometric language underlying superstring theory. Especially, super-analogs of algebraic curves and of their moduli spaces provide a mathematically solid background for many aspects of the theory.

Interlude: Homological algebra

Every mathematical object is, at least formally, a *set*, a collection of simpler entities of some kind called *elements*. Thus, a circle "is" the set (collection) of its points, a ring is the set of the functions that form it etc. This approach is still the mainstream of mathematical reasoning. A mathematician usually does not understand a construction unless it is formulated in such terms.

Now, there are two fundamentally different and dual ways of constructing new sets (mathematical objects) out of ones we have already.

One is by *conditions*, say describing a circle by the equation $x^2 + y^2 = 1$.

The other is by *parametrization*, i.e., by presenting an exhaustive list of all the data in the collection. For example, the same circle can be parametrized by $x = \cos(\alpha)$, $y = \sin(\alpha)$.

Homological algebra, in a wider sense, is the part of mathematics that studies the interplay between these two types of description. Often it is not possible to give two exactly matching descriptions of the same object, there is a "gap" between them: not every element satisfying the conditions is listed. This gap is formalized in the mathematical concept of *cohomology*.

The origins of homological algebra were in topology, the part of geometry that studies rough shapes of spaces invariant under deformations. To understand such structures, one considers *cycles*, geometric images with no boundary. Two cycles are called *homologous*, if their difference is a boundary. In such a way one can tell, for example, the difference between a sphere and a torus: in the torus we can have 1-dimensional cycles non-homologous to each other, unlike in the sphere. See Fig. 4. So here the conditions (vanishing of the boundary) and the lists (being a boundary) stem from the same geometric concept.

A mathematical structure allowing for systematic study of such phenomena (conditions vs. lists) is called

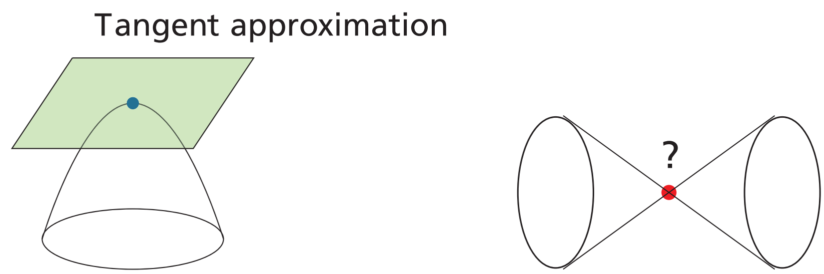


Figure 5: A smooth and a singular space.

a *cochain complex*, a vector space V together with a "differential" (analog of the boundary operator) d which, applied twice, gives 0. This formalizes the fundamental geometric property that "the boundary has no boundary." In most cases there is also a grading: vectors are assigned integer degrees.

This type of approach has also slowly gained ground in physics where it is known as the BRST quantization (named after physicists C. Becchi, A. Rouet, R. Stora and I. Tyitin who first introduced it in the 1970s). In this approach, only the "states" (vectors) ψ annihilated by d, are considered physical. Further, two physical states ψ, ψ' are considered equivalent (physically the same!) if their difference is of the form $d(\phi)$. Thus the actual physical meaning is assigned to the cohomology, i.e., to the gap! The full implications of this bold idea are still not fully understood.

Derived geometry

One can say that geometry teaches us how to pass from flat (linear) spaces to curved spaces (called manifolds). A manifold has a flat approximation associated to each point: the tangent space. A different choice of the concept of a flat space may be often upgraded to a curved generalization. The idea of derived geometry is to match this approach with that of homological algebra. That is, we consider, as flat models, not linear spaces but complexes as above.

The motivation for this was originally purely intrinsic to mathematics. It was known for a long time that moduli spaces (spaces of parameters of geometric structures) can be *singular*, i.e., possess points near which linear approximation breaks down, like the sharp point of a cone (see Fig. 5). It turns out that introduction of derived structures allows one to overcome these difficulties by producing new objects, which do possess nice linear approximations, but these approximations are complexes!

However, passing to the derived world drastically enlarges our supply of geometric objects. Along with "spaces" in the usual sense (even when understood as schemes in the sense of Grothendieck) one finds other types of geometric objects, some known, some new, which can be roughly classified by the range of the degrees of their tangent spaces (complexes) (see Fig. 6 where some of these types and the corresponding ranges are outlined). Thus, *stacks* (whose range comprises (-1) and 0) describe "spaces with internal symmetry" (similar to gauge symmetry in physics); most moduli spaces are known to be, in fact, stacks. Higher stacks describe even more sophisticated symmetries.

In some intuitive sense, the positive (right-hand) range corresponds to "geometry in the small", where we focus on small details near a complicated

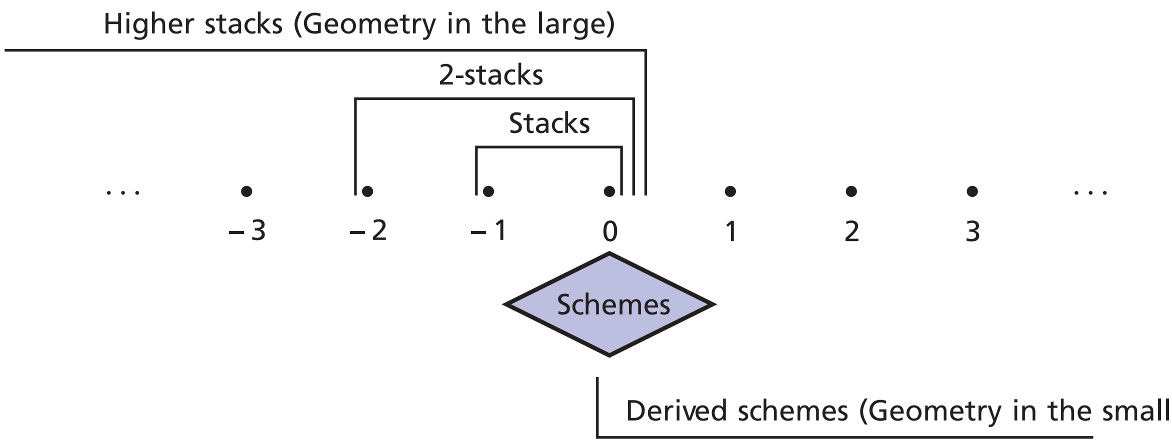


Figure 6: Panorama of derived geometry.

(singular) point of a geometric object. The negative (left hand) range corresponds similarly to "geometry in the large", where we focus on large scale, topological properties of spaces. It is remarkable that these two complementary aspects of geometry find a natural common framework.

Although somewhat abstract in its origins, derived geometry has by now found many remarkable applications in physics. Thus, derived stacks (objects belonging mostly to the right hand direction) provide the source of integration cycles in topological quantum field theories. Derived analogs of symplectic manifolds (geometric objects at the basis of Hamiltonian formalism of classical mechanics) have, in the last few years, emerged as a structure carried by many moduli spaces.

There is also a strong connection to super-geometry as the concept of commutativity in the derived world also involves the super sign rule (1). In fact, the sign system underlying (1) (such structures are called *Picard groupoids*) can be given a purely topological interpretation, in terms of the classification of mappings between higher-dimensional spheres (so-called stable homotopy groups). The degrees deg (f) (assumed integer) correspond to the integer invariant of maps between spheres of the same dimension, also known as the degree. The two signs $\pm$ corresponds to two types of maps between spheres of dimension $n+1$ and n, where $n = 3$ or more.

Conclusion

These are just a few examples of new geometric techniques that mathematicians use. Which type of geometry describes "the real world"? Which other types may be necessary for such description? So far, we do not know. The concept of space-time at extremely small distances may not even make sense as such, and some grainier, more chaotic quantum structure may replace it. But to be able to even talk about such things meaningfully, we need bridges connecting them to our geometric intuition and to our human patterns of thought. It is likely our lack of imagination of what kinds of geometry are possible that prevents us from asking the right questions.

To me, the fascinating power of super-geometry and the physical promise of supersymmetry suggest that commutativity in some even higher sense may open the doors to new geometric worlds relevant to physics. In particular, I think that structures related to stable homotopy groups of spheres, a classical subject of algebraic topology, may provide a guide to such new worlds.