Anisotropic universes in bigravity theory and homothetic metric solutions

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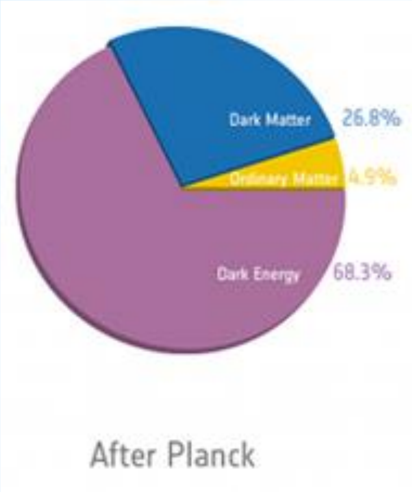
With *M. Volkov* (arXiv:1302.6198)

Dark Energy → cosmological constant Λ



$$\Lambda \leq 10^{-120} m_{PL}^2$$

Modified gravity

massive gravity

General Relativity → graviton: massless spin 2

Massive theory

Fierz-Pauli ghost-free linear theory

de Rham-Gabadadze-Tolley Non-linear extension

Hassan-Rosen Extension to bigravity

Question:

- Stability of FLRW universe
- de Sitter : attractor ? ⇒ **cosmic no hair conjecture**

In GR, if a cosmological constant exists,
it was proved by Wald for Bianchi models

- Bianchi universe (homogeneous but anisotropic spacetime)

KM, Volkov ('13) arXiv:1302.6198

- Homothetic solution is an attractor
- The shear density drops as a^{-3} (In GR a^{-6})
- Chaotic behaviour in Type IX

Basic equations

$$S[g, f, \mathrm{matter}] = \frac{1}{2\kappa_g^2} \int d^4x \sqrt{-g}\, R(g) + \frac{1}{2\kappa_f^2} \int d^4x \sqrt{-f}\, \mathcal{R}(f)$$

$$-\frac{m^2}{\kappa^2} \int d^4x \sqrt{-g}\, \mathscr{U}[g, f] + S_g^{[\mathrm{m}]}[g, \text{g-matter}] + S_f^{[\mathrm{m}]}[f, \text{f-matter}]$$

Interaction term

$$\kappa^2 = \kappa_g^2 + \kappa_f^2$$

$$\mathscr{U} = \sum_{k=0}^{4} b_k\, \mathscr{U}_k(\gamma), \qquad \gamma^\mu{}_\nu = \sqrt{g^{\mu\alpha} f_{\alpha\nu}} \qquad \lambda_A : \text{eigenvalues of } \gamma^\mu{}_\nu$$

$$\mathscr{U}_0(\gamma) = -\frac{1}{4!}\, \epsilon_{\mu\nu\rho\sigma}\epsilon^{\mu\nu\rho\sigma} = 1 \qquad \mathscr{U}_1(\gamma) = -\frac{1}{3!}\, \epsilon_{\mu\nu\rho\sigma}\epsilon^{\alpha\nu\rho\sigma}\gamma^\mu{}_\alpha = \sum_A \lambda_A$$

$$\mathscr{U}_2(\gamma) = -\frac{1}{2!}\, \epsilon_{\mu\nu\rho\sigma}\epsilon^{\alpha\beta\rho\sigma}\gamma^\mu{}_\alpha\gamma^\nu{}_\beta = \sum_{A<B} \lambda_A\lambda_B$$

$$\mathscr{U}_3(\gamma) = -\frac{1}{3!}\, \epsilon_{\mu\nu\rho\sigma}\epsilon^{\alpha\beta\gamma\sigma}\gamma^\mu{}_\alpha\gamma^\nu{}_\beta\gamma^\rho{}_\gamma = \sum_{A<B<C} \lambda_A\lambda_B\lambda_C$$

$$\mathscr{U}_4(\gamma) = -\frac{1}{4!}\, \epsilon_{\mu\nu\rho\sigma}\epsilon^{\alpha\beta\gamma\delta}\gamma^\mu{}_\alpha\gamma^\nu{}_\beta\gamma^\rho{}_\gamma\gamma^\sigma{}_\delta = \lambda_0\lambda_1\lambda_2\lambda_3$$

m : graviton mass

a flat space is a solution

$\Rightarrow$

$$b_0 = 4c_3 + c_4 - 6, \quad b_1 = 3 - 3c_3 - c_4,$$

$$b_2 = 2c_3 + c_4 - 1, \quad b_3 = -(c_3 + c_4), \quad b_4 = c_4.$$

Basic equations

$$G_{\mu\nu} = \kappa_g^2 \left[T^{[\gamma]}_{\mu\nu} + T^{[\mathrm{m}]}_{\mu\nu} \right] \qquad \mathcal{G}_{\mu\nu} = \kappa_f^2 \left[\mathcal{T}^{[\gamma]}_{\mu\nu} + \mathcal{T}^{[\mathrm{m}]}_{\mu\nu} \right]$$

$$\kappa_g^2 T^{[\gamma]\mu}{}_{\nu} = m_g^2 (\tau^{\mu}{}_{\nu} - \mathscr{U} \, \delta^{\mu}_{\nu}) \qquad \kappa_f^2 \mathcal{T}^{[\gamma]\mu}{}_{\nu} = -m_f^2 \frac{\sqrt{-g}}{\sqrt{-f}} \, \tau^{\mu}{}_{\nu} \,,$$

$$\tau^{\mu}{}_{\nu} = \{ b_1 \, \mathscr{U}_0 + b_2 \, \mathscr{U}_1 + b_3 \, \mathscr{U}_2 + b_4 \, \mathscr{U}_3 \} \gamma^{\mu}{}_{\nu} - \{ b_2 \, \mathscr{U}_0 + b_3 \, \mathscr{U}_1 + b_4 \, \mathscr{U}_2 \} (\gamma^2)^{\mu}{}_{\nu}$$
$$+ \{ b_3 \, \mathscr{U}_0 + b_4 \, \mathscr{U}_1 \} (\gamma^3)^{\mu}{}_{\nu} - b_4 \, \mathscr{U}_0 \, (\gamma^4)^{\mu}{}_{\nu}$$

$$\overset{(g)}{\nabla}_{\mu} \, T^{[\mathrm{m}]\mu}{}_{\nu} = 0, \qquad \overset{(f)}{\nabla}_{\mu} \, \mathcal{T}^{[\mathrm{m}]\mu}{}_{\nu} = 0$$

$$\overset{(g)}{\nabla}_{\mu} T^{[\gamma]\mu}{}_{\nu} = 0. \qquad \overset{(f)}{\nabla}_{\mu} \mathcal{T}^{[\gamma]\mu}{}_{\nu} = 0$$

Homothetic metrics: $f_{\mu\nu} = K^2\, g_{\mu\nu}$ $\Rightarrow$ $\gamma^\mu{}_\nu = K\,\delta^\mu{}_\nu$

$$\kappa_g^2 T^\mu{}_\nu = -\Lambda_g(K)\delta^\mu{}_\nu\,,\ \ \kappa_f^2 \mathcal{T}^\mu{}_\nu = -\Lambda_f(K)\delta^\mu{}_\nu$$

$$\Lambda_g(K) = m_g^2\left(b_0 + 3b_1\,K + 3b_2\,K^2 + b_3\,K^3\right)$$

$$\Lambda_f(K) = m_f^2\left(b_1/K^3 + 3b_2/K^2 + 3b_3/K + b_4\right)$$

$$\overset{(g)}{\nabla}_\mu T^{[\gamma]\mu}{}_\nu = 0. \qquad \overset{(f)}{\nabla}_\mu \mathcal{T}^{[\gamma]\mu}{}_\nu = 0$$ $\Rightarrow$ K: constant

GR with a cosmological constant

$$G_{\mu\nu} + \Lambda_g g_{\mu\nu} = \kappa_g^2 T^{[\mathrm{m}]}_{\mu\nu}$$

$\Lambda_g = K^2\Lambda_f$: quartic equation for K

$$\mathcal{T}^{[\mathrm{m}]}_{\mu\nu} = K^2 T^{[\mathrm{m}]}_{\mu\nu}$$

Bianchi Spacetimes

$$[\xi_a, \xi_b] = C^c{}_{ab}\xi_c \qquad \xi_a : \text{Killing vectors}$$

$$C^c{}_{ab} = n^{cd}\epsilon_{dab} + a(\delta^1_a\delta^c_b - \delta^1_b\delta^c_a) \qquad n^{ab} = \mathrm{diag}[n^{(1)}, n^{(2)}, n^{(3)}]$$

$a = 0$ ⇒ Class A

	I	II	VI_0	VII_0	VIII	IX
$n^{(1)}$	0	1	1	1	1	1
$n^{(2)}$	0	0	-1	1	1	1
$n^{(3)}$	0	0	0	0	-1	1

$$ds_g^2 = -\alpha^2 dt^2 + e^{2\Omega}e^{2\beta_{ij}}\omega_i\omega_j \qquad ds_f^2 = -\mathcal{A}^2 dt^2 + e^{2\mathcal{W}}e^{2\mathcal{B}_{ij}}\omega_i\omega_j$$

$$(\beta_{ij}) = \begin{pmatrix} \beta_+ + \sqrt{3}\beta_- & 0 & 0 \\ 0 & \beta_+ - \sqrt{3}\beta_- & 0 \\ 0 & 0 & -2\beta_+ \end{pmatrix}$$

$$(\mathcal{B}_{ij}) = \begin{pmatrix} \mathcal{B}_+ + \sqrt{3}\mathcal{B}_- & 0 & 0 \\ 0 & \mathcal{B}_+ - \sqrt{3}\mathcal{B}_- & 0 \\ 0 & 0 & -2\mathcal{B}_+ \end{pmatrix}.$$

Bianchi I

$$S = \frac{3}{\kappa_g^2}\int d^4x \frac{e^{3\Omega}}{\alpha}\left(-\dot{\Omega}^2+\dot{\beta}_+^2+\dot{\beta}_-^2\right) + \frac{3}{\kappa_f^2}\int d^4x\,\frac{e^{3\mathcal{W}}}{\mathcal{A}}\left(-\dot{\mathcal{W}}^2+\dot{\mathcal{B}}_+^2+\dot{\mathcal{B}}_-^2\right) - \frac{m^2}{\kappa^2}\int d^4x(\alpha V_g+\mathcal{A}\mathcal{V}_f)$$

$$V_g = -\Big[b_0e^{3\Omega}+b_3e^{3\mathcal{W}}+b_1e^{\mathcal{W}+2\Omega}\left(e^{-2(\mathcal{B}_+-\beta_+)}+2e^{\mathcal{B}_+-\beta_+}\cosh[\sqrt{3}(\mathcal{B}_--\beta_-)]\right) + b_2e^{2\mathcal{W}+\Omega}\left(e^{2(\mathcal{B}_+-\beta_+)}+2e^{-(\mathcal{B}_+-\beta_+)}\cosh[\sqrt{3}(\mathcal{B}_--\beta_-)]\right)\Big]$$

$$\mathcal{V}_f = -\Big[b_1e^{3\Omega}+b_4e^{3\mathcal{W}}+b_2e^{\mathcal{W}+2\Omega}\left(e^{-2(\mathcal{B}_+-\beta_+)}+2e^{\mathcal{B}_+-\beta_+}\cosh[\sqrt{3}(\mathcal{B}_--\beta_-)]\right) + b_3e^{2\mathcal{W}+\Omega}\left(e^{2(\mathcal{B}_+-\beta_+)}+2e^{-(\mathcal{B}_+-\beta_+)}\cosh[\sqrt{3}(\mathcal{B}_--\beta_-)]\right)\Big]$$

perfect fluid $\rho_g^{(\mathrm{m})},\quad P_g^{(\mathrm{m})} \qquad \rho_f^{(\mathrm{m})},\quad P_f^{(\mathrm{m})}.$

$$\frac{1}{2}\dot{\Omega}^2 = \frac{1}{2}\left(\dot{\beta}_+^2 + \dot{\beta}_-^2\right) + \frac{m_g^2}{6}\alpha^2 e^{-3\Omega} V_g + \frac{\alpha^2\kappa_g^2}{6}\rho_g^{(\mathrm{m})}$$

$$\ddot{\Omega} - \frac{\dot{\alpha}}{\alpha}\dot{\Omega} + 3\dot{\Omega}^2 = \frac{m_g^2}{6}\alpha e^{-3\Omega}\left[\alpha\left(3V_g + \frac{\partial V_g}{\partial \Omega}\right) + \mathcal{A}\frac{\partial \mathcal{V}_f}{\partial \Omega}\right] + \frac{\alpha^2\kappa_g^2}{2}\left(\rho_g^{(\mathrm{m})} - P_g^{(\mathrm{m})}\right)$$

$$\ddot{\beta}_\pm - \frac{\dot{\alpha}}{\alpha}\dot{\beta}_\pm + 3\dot{\Omega}\dot{\beta}_\pm = -\frac{m_g^2}{6}\alpha e^{-3\Omega}\left(\alpha\frac{\partial V_g}{\partial \beta_\pm} + \mathcal{A}\frac{\partial \mathcal{V}_f}{\partial \beta_\pm}\right)$$

$$\frac{1}{2}\dot{\mathcal{W}}^2 = \frac{1}{2}\left(\dot{\mathcal{B}}_+^2 + \dot{\mathcal{B}}_-^2\right) + \frac{m_f^2}{6}\mathcal{A}^2 e^{-3\mathcal{W}}\mathcal{V}_f + \frac{\mathcal{A}^2\kappa_f^2}{6}\rho_f^{(\mathrm{m})}$$

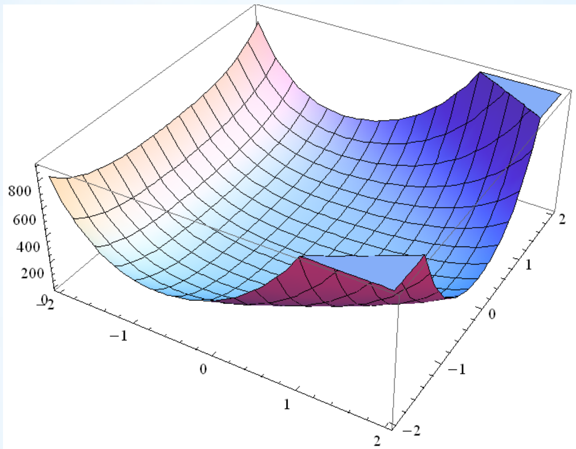
$$\ddot{\mathcal{W}} - \frac{\dot{\mathcal{A}}}{\mathcal{A}}\dot{\mathcal{W}} + 3\dot{\mathcal{W}}^2 = \frac{m_f^2}{6}\mathcal{A}e^{-3\mathcal{W}}\left[\alpha\frac{\partial V_g}{\partial \mathcal{W}} + \mathcal{A}\left(3\mathcal{V}_f + \frac{\partial \mathcal{V}_f}{\partial \mathcal{W}}\right)\right] + \frac{\mathcal{A}^2\kappa_f^2}{2}\left(\rho_f^{(\mathrm{m})} - P_f^{(\mathrm{m})}\right)$$

$$\ddot{\mathcal{B}}_\pm - \frac{\dot{\mathcal{A}}}{\mathcal{A}}\dot{\mathcal{B}}_\pm + 3\dot{\mathcal{W}}\dot{\mathcal{B}}_\pm = -\frac{m_f^2}{6}\mathcal{A}e^{-3\mathcal{W}}\left(\alpha\frac{\partial V_g}{\partial \mathcal{B}_\pm} + \mathcal{A}\frac{\partial \mathcal{V}_f}{\partial \mathcal{B}_\pm}\right)$$

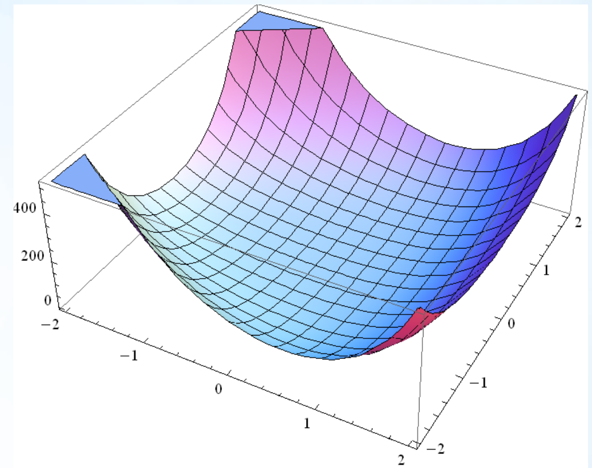
$$m_g^2 = \frac{m^2\kappa_g^2}{\kappa^2}, \quad m_f^2 = \frac{m^2\kappa_f^2}{\kappa^2}$$

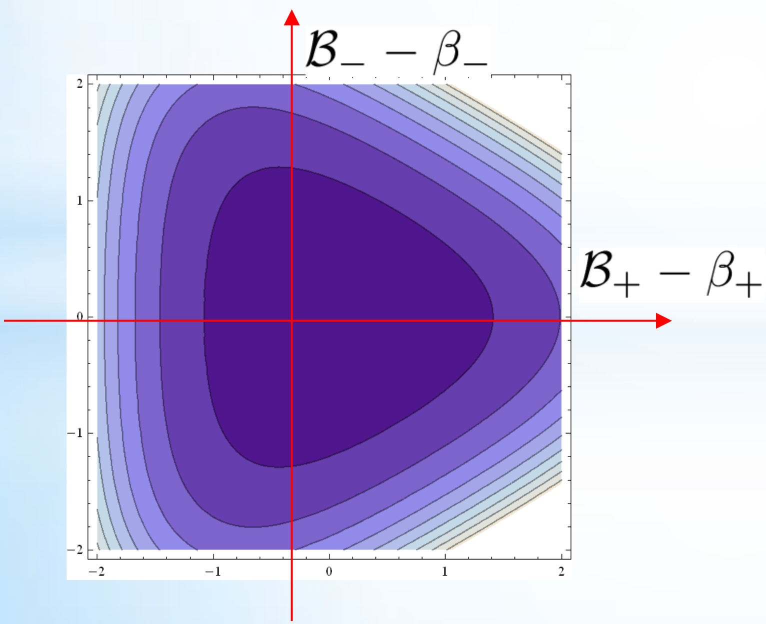
$b_1, b_2, b_3 < 0$

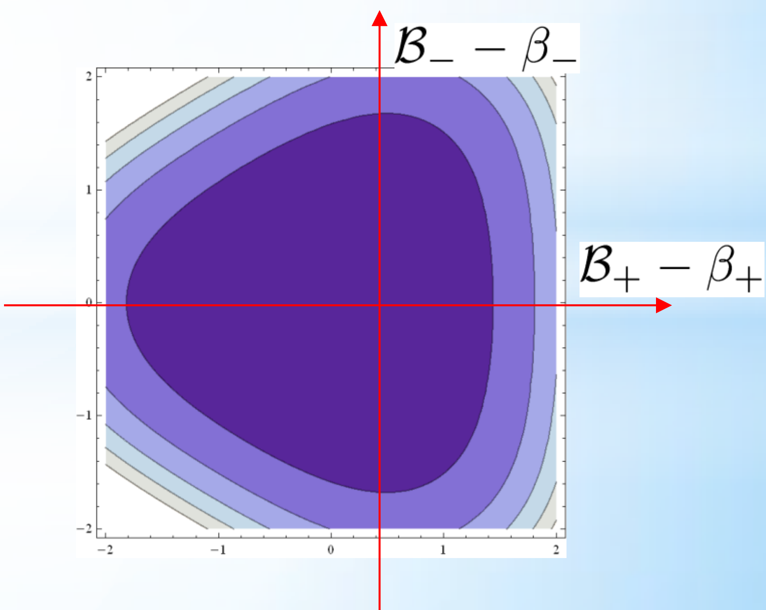
V_g



$\mathcal{V}_f$







$$\mathcal{B}_\pm = \beta_\pm \quad \Rightarrow \quad e^{3\Omega}\,\frac{\dot\beta_\pm}{\alpha} = \sigma_{\pm(0)}\,, \qquad e^{3\mathcal{W}}\,\frac{\dot{\mathcal{B}}_\pm}{\mathcal{A}} = \mathcal{S}_{\pm(0)}$$

$$e^{3(\mathcal{W}-\Omega)}\,\frac{\alpha}{\mathcal{A}} = \frac{\mathcal{S}_{+(0)}}{\sigma_{+(0)}} = \frac{\mathcal{S}_{-(0)}}{\sigma_{-(0)}} \equiv C^2$$

Hamiltonian constraints + EOM ⇨

$$\left[\alpha\left(e^{\mathcal{W}}\right)^{\cdot} - \mathcal{A}\left(e^{\Omega}\right)^{\cdot}\right]\left(b_1 + 2b_2 e^{\mathcal{W}-\Omega} + b_3 e^{2(\mathcal{W}-\Omega)}\right) = 0$$

$$(1)\ \alpha\left(e^{\mathcal{W}}\right)^{\cdot} - \mathcal{A}\left(e^{\Omega}\right)^{\cdot} = 0.$$

$$\Rightarrow \quad f_{\mu\nu} = C^2\, g_{\mu\nu}$$ homothetic $C = K$

$$(2)\ b_1 + 2b_2 e^{\mathcal{W}-\Omega} + b_3 e^{2(\mathcal{W}-\Omega)} = 0 \qquad e^{\mathcal{W}-\Omega} = \xi_0$$

$$\mathcal{A} = \sqrt{\frac{\Lambda_g(\xi_0)}{\Lambda_f(\xi_0)}}\,\alpha$$

$$e^{\mathcal{W}} = \xi_0\, e^{\Omega}$$

$$\rho_f = \frac{\Lambda_f(\xi_0)}{\Lambda_g(\xi_0)}\,\rho_g$$

homothetic solution=vacuum Bianchi I with a cosmological constant Λ in GR

analytic solution

$\Lambda > 0$

$$e^{\Omega} = \frac{1}{2^{1/3}} e^{\pm H_0(t-t_0)} \left(1 - \frac{\sigma_0^2}{H_0^2} e^{\mp 6H_0(t-t_0)}\right)^{1/3}$$

$$H_0 = \sqrt{\Lambda/3}$$

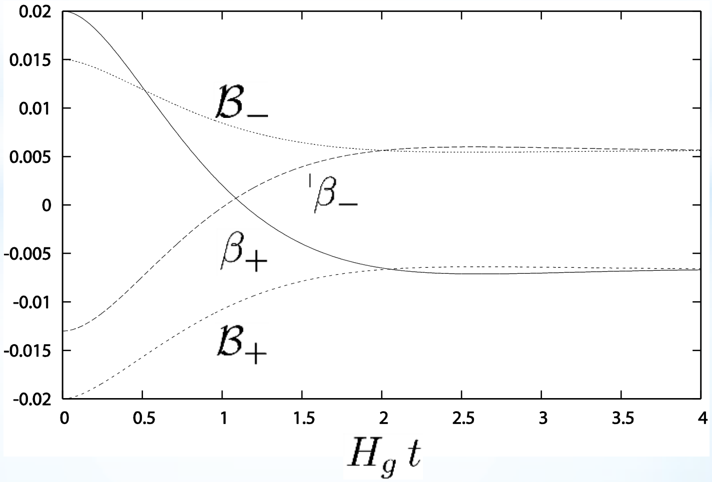
$$e^{\beta_\pm} = e^{\beta_{\pm(0)}} \left(\frac{1 - \frac{\sigma_0}{H_0} e^{\mp 3H_0(t-t_0)}}{1 + \frac{\sigma_0}{H_0} e^{\mp 3H_0(t-t_0)}}\right)^{\pm \frac{\sigma_\pm^{(0)}}{3\sigma_0}}$$

$$\sigma_0^2 = \sigma_{+(0)}^2 + \sigma_{-(0)}^2$$

$$\Sigma^2 = \frac{\sigma^2}{H^2} \propto e^{-6\Omega}$$

Homothetic solution is an attractor in Bianchi I

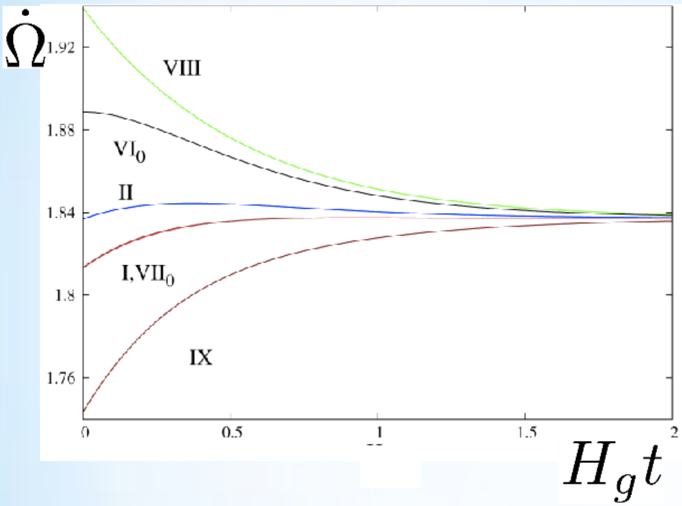
$$\mathcal{B}_\pm - \beta_\pm \to 0$$

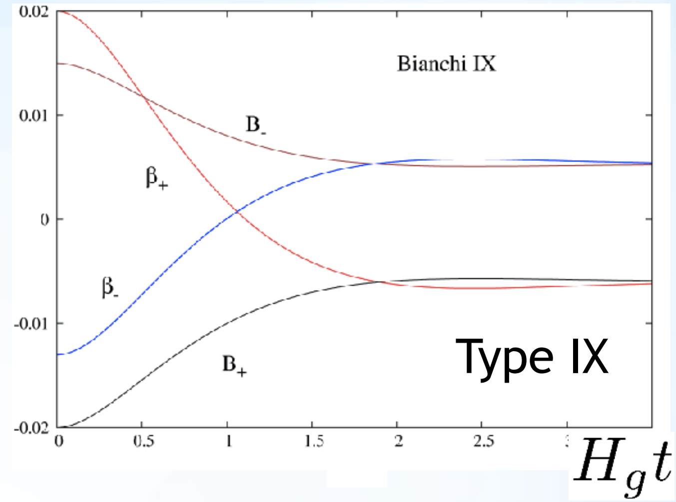


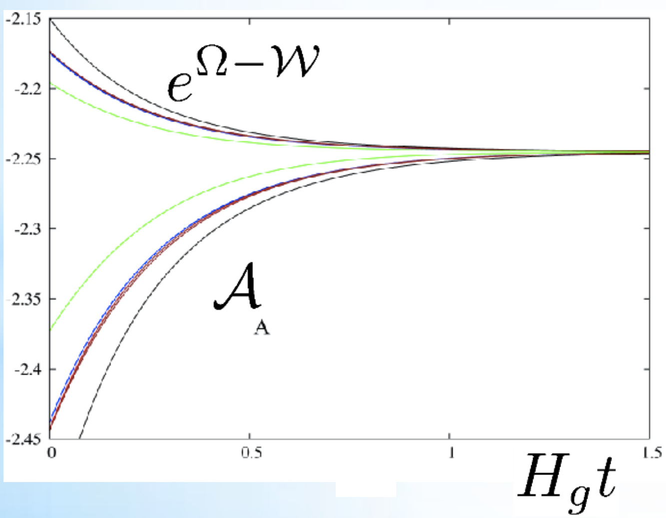
$$H_g = \sqrt{\Lambda_g/3}$$

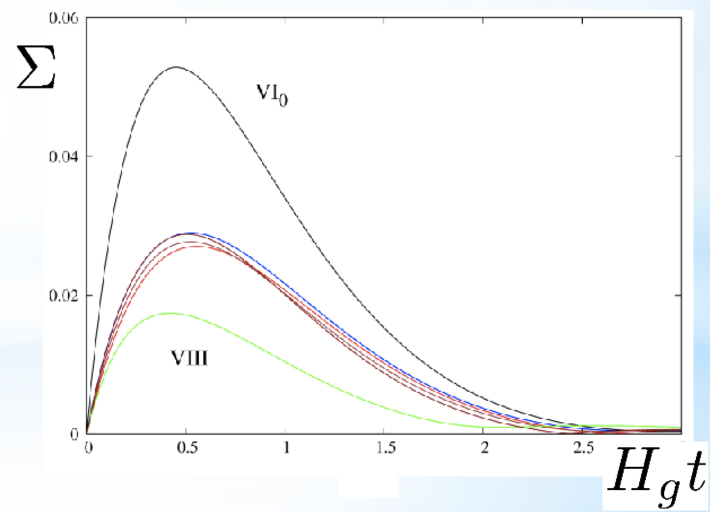
More General Bianchi Types

Numerical Results $c_3 = 1\,, \quad c_4 = 0.3\,, \quad m_g = 0.54m\,, \quad m_f = 0.84m$









$\Rightarrow$ Approach to homothetic metrics homothetic solution $\Rightarrow$ GR with a cosmological constant

Shear drops fast as $\sigma^2 \sim \dot{\beta}_+^2 + \dot{\beta}_-^2 \sim e^{-6\Omega}$

However, it does not drop so fast:

$$\sigma^2 \sim \dot{\beta}_+^2 + \dot{\beta}_-^2 \sim e^{-3\Omega}$$

This is the same as matter fluid

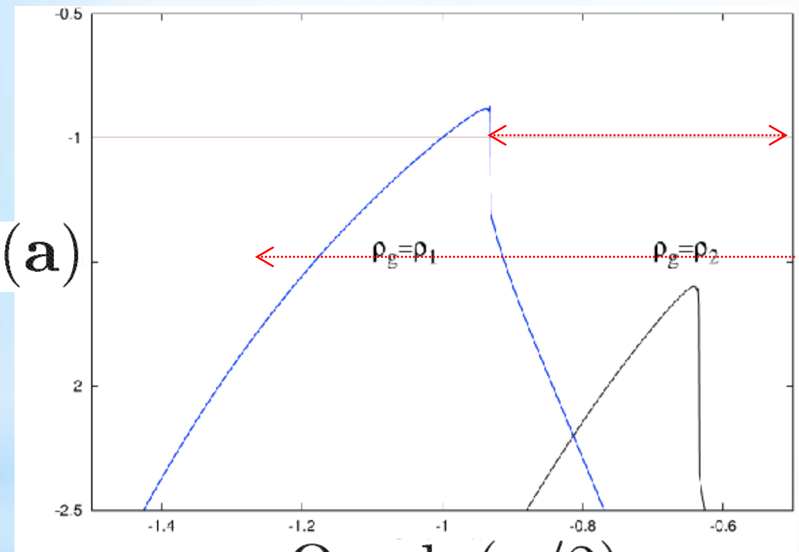
Any Observational Effect ?

de Sitter spacetime is not always an attractor

Closed FLRW universe

$$\dot{\mathbf{a}}^2 + V(\mathbf{a}) = -k \quad (k = 1)$$

$V(\mathbf{a})$



$\Omega = \ln(\mathbf{a}/2)$

$$\rho_1/m^4 = 0.1 \times e^{-4\Omega}$$

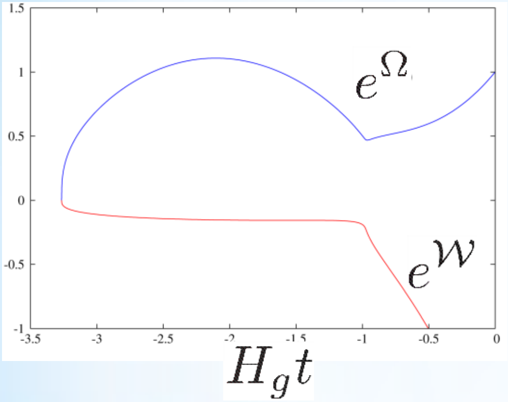
bounce -> de Sitter

$$\rho_2/m^4 = 0.25 \times e^{-4\Omega} + 0.25 \times e^{-3\Omega}$$

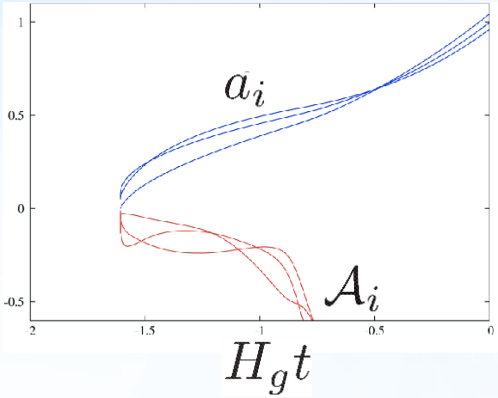
collapse -> singularity

Initial Stage (near singularity)

Bianchi I

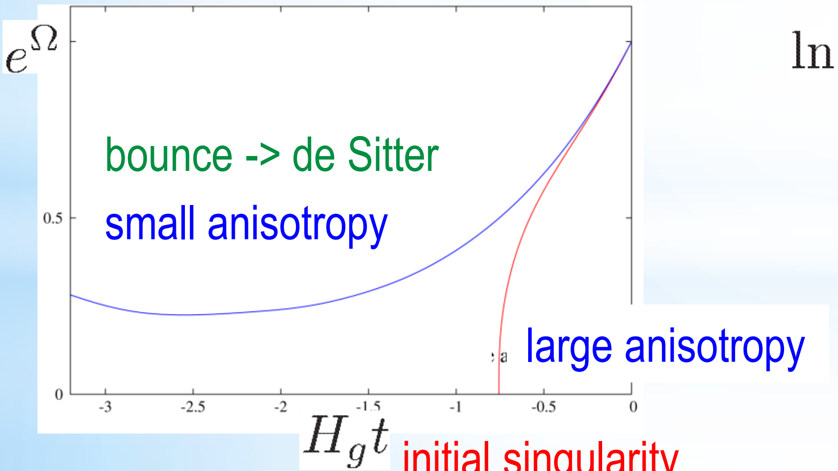


Bianchi IX

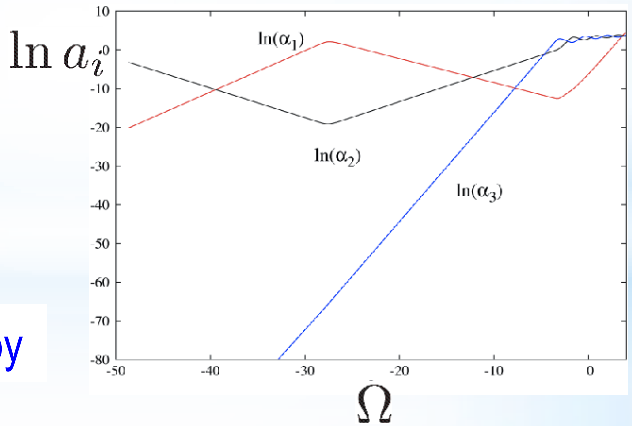


Three spatial components of metrics

vacuum Bianchi IX



initial singularity



chaotic behaviour near singularity

perturbations around a homothetic solution

$$g_{\mu\nu} = g^{(0)}_{\mu\nu} + \epsilon h_{\mu\nu}\,, \quad f_{\mu\nu} := K^2 \tilde{f}_{\mu\nu} = K^2 \left(g^{(0)}_{\mu\nu} + \epsilon k_{\mu\nu} \right)$$

$$\psi_{\mu\nu} := m_f^2 h_{\mu\nu} + K^2 m_g^2 k_{\mu\nu}$$

$$\varphi_{\mu\nu} := h_{\mu\nu} - k_{\mu\nu}$$

$$\bar{\psi}_{\mu\nu} := \psi_{\mu\nu} - \frac{1}{2}\psi g^{(0)}_{\mu\nu}$$

$$\bar{\varphi}_{\mu\nu} := \varphi_{\mu\nu} - \frac{1}{2}\varphi g^{(0)}_{\mu\nu}$$

de Sitter background

$$\overset{(0)}{\Box}\bar{\psi}^{(\mathrm{TT})}_{\mu\nu} - \frac{2}{3}\Lambda_g \bar{\psi}^{(\mathrm{TT})}_{\mu\nu} = 0$$

massless mode

$$\overset{(0)}{\Box}\bar{\varphi}_{\mu\nu} - \left(\frac{2}{3}\Lambda_g + m_{\mathrm{eff}}^2 \right) \bar{\varphi}_{\mu\nu} = 0 \qquad 2\overset{(0)}{\nabla}{}^{\mu}\bar{\varphi}_{\mu\nu} - \overset{(0)}{\nabla}{}_{\nu}\bar{\varphi} = 0 \qquad \bar{\varphi} = 0$$

massive mode

$$m_{\mathrm{eff}}^2 = \left(m_g^2 + \frac{m_f^2}{K^2} \right) (b_1 K + 2b_2 K^2 + b_3 K^3)$$

⇨ stable

Higuchi bound $m_{\mathrm{eff}}^2 = 2\Lambda_g/3 = 2H_g^2$ partially massless

general vacuum homothetic background

$$\overset{(0)}{\Box}\bar{\psi}^{(\mathrm{TT})}_{\mu\nu} + 2\overset{(0)}{C}{}_{\mu}{}^{\alpha}{}_{\nu}{}^{\beta}\bar{\psi}^{(\mathrm{TT})}_{\alpha\beta} - \frac{2}{3}\Lambda_g \bar{\psi}^{(\mathrm{TT})}_{\mu\nu} = 0$$

$$\overset{(0)}{\Box}\bar{\varphi}_{\mu\nu} + 2\overset{(0)}{C}{}_{\mu}{}^{\alpha}{}_{\nu}{}^{\beta}\bar{\varphi}_{\alpha\beta} - \left(\frac{2}{3}\Lambda_g + m_{\mathrm{eff}}^2 \right) \bar{\varphi}_{\mu\nu} = 0$$

Summary

- We discuss anisotropic Bianchi class A spacetimes in bigravity

- We present the analytic solution for homotheic Bianchi I model

- We show

  The homothetic solution is an attractor if anisotropy is small

  ⟹ cosmic no hair

  The shear energy density drops as matter density

  The chaotic behaviour of early stage of Bianchi IX model

Further Questions:

- More generic anisotropic universes

  Bianchi class B   Wainwright's approach

  "inhomogeneous" (non-bidiagonal) case

- Homothetic solution is also an attractor
  for Inhomogeneous spacetimes ?

  Cosmic no hair conjecture ?

  Black hole uniqueness ?

Thank you for your attention