Lorentz or spatial Weyl invariance? A tour through Shape Dynamics.

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Outline

Some quotes

"A few decades ago it seemed quite certain that one had to express the whole of physics in four-dimensional form. But now it seems that four-dimensional symmetry is not of such overriding importance, since the description of nature sometimes gets simplified when one departs from it." - Dirac (1963)

"It should be noted that 'relativity of simultaneity' is contingent on assuming that a measuring rod or clock being set in motion or brought to rest does not change the length of the rod or the rate of the clock. "- Poincaré (1909)

"An increasing amount of evidence shows that the true dynamical degrees of freedom of the gravitational field can be identified directly with the conformally invariant geometry of three-dimensional spacelike hypersurfaces embedded in space-time."- York (1972)

A brief history of classical scale-invariance in GR.

- **Weyl (1919):** Introduced a scale-connection A_μ through minimal coupling, asking why should the direction of a vector have no absolute meaning, but its length have one? This, he says, "has no factual basis, and only seems to be due to the derivation of the theory from the flat one"
- **Brans-Dicke (1960):** "It is evident that two rods side by side, stationary with respect to each other, can be intercompared....this cannot be done for....rods with either a space- or time-like separation". [...] a hydrogen atom on Sirius has the same diameter as one on the Earth...is either a definition or else meaningless".

And a lot more, Dirac, Bekenstein, etc.

A brief history of scale-invariance in ADM (3+1).

- **York (1971):** Solves the initial value problem for GR by exploring Weyl transformations *of the 3-metric* and *using constant mean curvature gauge.*

- **Barbour et al (2005):** Rederived York's initial value problem for closed spatial manifolds, with the correct scaling, from a variational principle using "volume-preserving Weyl transformations". I.e. Weyl transformations that preserve the *total* volume of the Universe.

Here not so much history.

Outline

Initial value problem of GR

Spacetime: $\Sigma \times \mathbb{R}$. ADM-decomposition of the space-time metric:

$$ds^2 = -N^2 dt^2 + g_{ab}(dx^a + \xi^a\, dt)(dx^b + \xi^b\, dt)$$

$$S_{\mathrm{EH}}[{}^4 g_{\mu\nu}] \to S_{\mathrm{EH}}[(g_{ab}, \dot{g}_{ab}, N, \xi^a)].$$

Legendre transform $(g_{ab}, \dot{g}_{ab}) \to (g_{ab}, \pi^{ab})$ yields constraints:

- $$S(x) = \left(\frac{\pi^{ab}\pi_{ab} - \frac{1}{2}\pi^2}{|g|} - (R - 2\Lambda)\right)(x) = 0$$
  - Generates time refoliations.
- $$H(x) = \pi^{ab}_{\ ;a}(x) = 0$$
  - Generates spatial diffeomorphisms.

York's approach

York: under a Weyl tranformation the metric transforms as $g_{ab} \to \Omega^4 g_{ab}$

Scalar constraint $S(x)$ gives 2nd order PDE for Ω.

Assuming that $g_{ab}\pi^{ab} = c\sqrt{g}$:

- Can always solve PDE for spatial scale $\Omega = \Omega[c, \pi^{ab}, g_{ab}]$
- Given an initial (g_{ab}, π^{ab}), with π const. find Weyl related data that solves the scalar constraint. Only generic way to find valid initial data.

Conformal ADM and CMC gauge-fixing

What is the CMC gauge-fixed version of ADM? Conformal ADM in CMC.

Decompose $(g_{ab}, \pi^{ab}) \to (|g|^{-1/3} g_{ab}, |g|^{1/3}(\pi^{ab} - \frac{1}{3}\pi g^{ab}), |g|, \frac{2\pi}{3\sqrt{|g|}})$.

Imposing $\pi = c\sqrt{g}$ and using the York results $\Rightarrow$ phase space reduction to Weyl invariant part. Problems:

- Requires background metric to separate out the scale part. Ok.
- Hard to work with reduced phase space. How to connect to geometry? Reconstructed (g_{ab}, π^{ab}) require the scale (non-local). Not so ok.

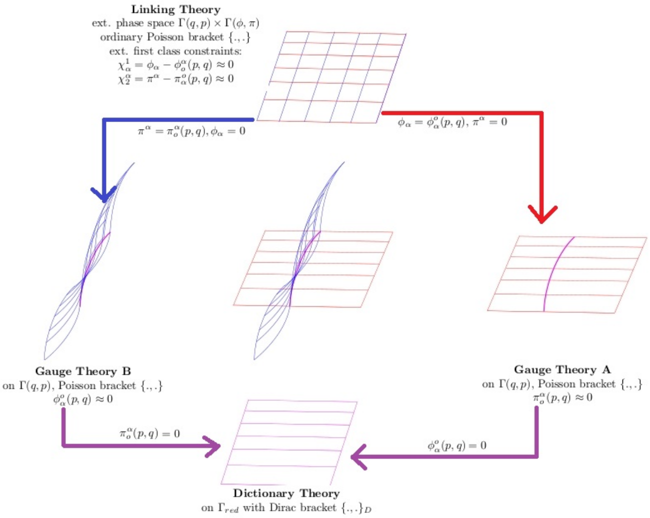
Q: What is Shape Dynamics?

What is Shape Dynamics

It's a Hamiltonian theory of gravity (3+1), about evolving spatial metric fields and its canonical momenta. Not a spacetime perspective.

- What is special about it is that it has inbuilt *spatial scale invariance.* Namely, it is invariant under dilatations: $(g_{ab}, \pi^{ab}) \rightarrow (\alpha g_{ab}, \frac{1}{\alpha}\pi^{ab})$. Dual role of π:
  - As the gauge fixing required for the York mechanism.
  - As a generator of Weyl transformations $\{g_{ab}, \pi(\epsilon)\} = \epsilon g_{ab}$, $\{\pi^{ab}, \pi(\epsilon)\} = -\epsilon\pi^{ab}$
- No space-time coordinate invariance. Refoliations "traded" for dilatations. Same number of degrees of freedom. No scalar mode problem.
- And it matches GR in a very broad set of circumstances! But not always...

How is Shape Dynamics constructed? Figure.



How is it constructed? Words.

1. Introduce artificial scalar fields (Stuckelberg) and extend ADM phase space: $(g, \pi) \to (g, \pi, \hat{\phi}, \pi_\phi)$

   where $\hat{\phi}$ is a Weyl factor that preserves the total volume of the Universe.
2. Extra constraint arises: $\pi_\phi - 4(\pi - \sqrt{g}\,\langle\pi\rangle) \approx 0$. Generates trivial symmetry.
3. Two different natural gauge fixings of extended ADM:
   - $\phi = 0$ **gauge fixing**. Gauge fixes extra constraint $\Rightarrow$ original ADM.
   - $\pi_\phi = 0$ **gauge fixing**. Partially gauge fixes extended scalar constraint.

   Reduced theory with local constraints generating diffeomorphisms and 3d Weyl transformations, and one global scalar constraint. Unfreeze global constraint and obtain H_{SD}.

Important: Both gauge fixings get back to original phase space (g, π) with canonical Poisson bracket.

A different perspective

A different perspective

Conformal ADM in CMC shows that gravity can also be naturally posed as a 3d Weyl invariant theory. It shows that there is the possibility of a different paradigm for gravity, not based on the geometry of space-time, but on the conformal geometry of space.

Yet conformal ADM in CMC requires non-local kinematics.

Two different natural "unfixings" to obtain a local theory, ADM or Shape Dynamics:

- Shape Dynamics acquires Weyl gauge symmetry.
- ADM acquires refoliation symmetry.

$\Rightarrow$ Local Lorentz invariance is only one possibility. There is another aesthetic, practical and fruitful perspective available.

Shape Dynamics vs Conformal ADM in CMC

How is Shape Dynamics different from a gauge-fixing of ADM? Not so much.

- Whenever $H_{\text{SD}} = 0$ and there exists a complete CMC gauge-fixing of spacetime, Shape Dynamics will have same physical observables as ADM.

Advantages of Shape Dynamics:

- **vs ADM**: No "problem of time" issues; global evolution operator, preferred "surfaces of simultaneity"; all constraints linear in the momenta (important for quantization).
- **vs Conformal ADM in CMC**: Local kinematics; in canonical variables (g, π); Shape Dynamics Hamiltonian is defined throughout phase space (off-shell), not just on intersection of constraint surfaces.

Where can we see these differences? Classical

- Since $H_{\rm gl} \not\approx 0$ easy to find initial data of Shape Dynamics not gauge-equivalent to initial data of ADM.
- To construct space-time from Shape Dynamics we need to rebuild lapse $N = \sqrt{-g_{00}}$. When CMC foliation exists, it is (generically) unique, $N_o[g_{ab}, \pi^{ab}; x)$.
  - Obstruction: CMC lapse breakdown, causing loss of equivalence from both sides.
  - Breakdown is irrelevant from the Shape Dynamics side. E.g. isotropic black hole is the closest to a "spacetime view" of a Shape Dynamics solution.

Additionally, new techniques available to address old problems. We will discuss a few.

Where can we see these differences? Quantization.

- Constraints linear in momenta: $\pi \approx 0$, $\pi^{ab}_{;a} \approx 0$.
  - Well defined geometric constraints on the wave function. True algebra of constraint functions.
  - Quantization doesn't involve $\frac{\delta^2}{\delta g_{ab}(x)\delta g_{ab}(x)}$ (ill defined).
- Different effective field theory (and modified gravity) approach: terms should respect spatial Weyl and diffeos as opposed to spacetime diffeos. E.g.
  - $S = \int d^3x \left(a\sqrt{\sigma^{ab}\sigma_{ab}} + CS(\Gamma_{\text{L.C.}})\right)$
- No traditional "problem of time". Hamiltonian no longer a constraint: $\hat{H}_{\text{SD}}\Psi[g] = \frac{d}{dt}\Psi[g]$
- Woodard & Tsamis (86) indicate no Weyl anomaly for closed case. If true:
  - $\pi - \langle\pi\rangle\sqrt{g} \approx 0 \Rightarrow g_{ab}(x)\frac{\delta\psi}{\delta g_{ab}(x)} = \frac{1}{V}\int d^3x' g_{ab}(x')(\frac{\delta\psi[g]}{\delta g_{ab}(x')})$.

Take away message from SD.

ADM (g_{ab}, π^{ab})

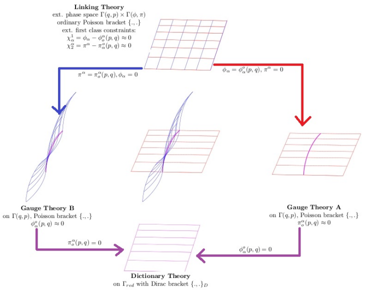
Local symmetries:

- 3-diffeomorphisms
- refoliations

Shape Dynamics (g_{ab}, π^{ab})

One Hamiltonian + local symmetries:

- 3-diffeomorphisms
- 3d-Weyl transformations



Shape Dynamics is to Conformal ADM in CMC

as Electromagnetism is to transverse gauge of vector potential.

Outline

General remarks

Relevance for AdS/CFT

Natural setting for studying such dualities: Weyl-invariant theory of the $d-1$ dimensional metric variables (bulk-bulk duality)

- Hamiltonian holography: Line element of the form $ds^2 = dt^2 - g_{ab}\, dx^a dx^b$. Asymptotic homogeneity of $\pi/\sqrt{|g|}$ and scalar curvature R.
- These two conditions imply asymptotic CMC lapse solution given by $N=1$:
  $\Rightarrow$ specific gauge that manifestly coincides with Shape Dynamics.
- Asymptotic Weyl symmetry in g_{ab} identified with Weyl symmetry of Shape Dynamics. However, homogeneous lapse does not propagate CMC condition into bulk. Manifest asymptotic equivalence lost in the bulk.

Large Volume expansion

Ansatz: $H_{SD} = \sum_{n=0}^{\infty} (\frac{V}{V_o})^{-\frac{2n}{3}} h_n$ and $\hat{\Omega} = \sum_{n=0}^{\infty} (\frac{V}{V_o})^{-\frac{2n}{3}} \omega_n$

large CMC-volume Hamiltonian

$$H_{SD} = (2\Lambda - \tfrac{1}{6}\langle\pi\rangle^2) - \langle R_o\rangle(\tfrac{V}{V_o})^{-\frac{2}{3}} + \langle\tfrac{\sigma^a_b\sigma^b_a}{|g|}\rangle(\tfrac{V}{V_o})^{-2} + \dots$$

$R_o[g]$ is in Yamabe gauge, i.e. R_o is homogeneous

Observations

1. asymptotic freezing of shape deg. of freedom
2. Using Hamilton-Jacobi formalism, do a volume expansion on the Jacobi functional.
   - We derive the correct form for Weyl anomalies (terms of the Jacobi functional that don't have volume dependence) up to term $R^2 - \frac{8}{3}R^{ab}R_{ab}$
3. Applicability: generic large CMC volume regime

Outline

What is encoded at the asymptotic boundary

- In vacuum, the GR action vanishes on-shell. All that remains are the boundary terms.
- The asymptotic boundary terms define linear and angular momenta, and the energy of the Universe:
  - Energy: equivalent to change in action by infinitesimal time-displacement of boundary
  - Momenta: infinitesimal linear spatial displacement of boundary.
  - These displacements, $N^\mu_{(i)}$, are ones that maintain the fall-off conditions: $\delta_{N^\mu_{(i)}} g_{ij} \sim 1/r$ and give finite energy, momentum, etc.
  - By restricting the general displacements in the full 3+1 diffeomorphism algebra to the asymptotically valid ones $N^\mu_{(i)}$ $[N^\mu, \tilde{N}^\nu] \to [N^\mu_{(i)}, N^\nu_{(j)}]$, we get the Poincaré algebra.

$\Rightarrow$ Poincaré is a remnant of the larger group of 4D diffeomorphisms.

What we get for Shape Dynamics

- Also the Poincaré algebra. But in a non-trivial way:

Remember that we "traded" ability to redefine time coordinates for ability to redefine local spatial scale?

- For asymptotically flat, unable to trade everything: a set of 4 generators of time displacements still asymptotically preserve maximal slicing, $N_o \sim \{1, x^a\}$.
- Exactly the ones we need to represent asymptotic boosts and time translations in Shape Dynamics asymptotic algebra. This allows us to obtain Poincaré invariance for a flat solution for Shape Dynamics.

More: $\boxed{\text{the ``ADM mass'', or energy, obtained is Weyl invariant}}$

$$\boxed{E_{\text{SD}}[g,\pi] = -\int_{\partial\Sigma} d^2y(2(k-k_o) + 8r_e\Omega^{;e})} \quad \text{Weyl invariant}$$

Outline

Work in progress: Equations of motion of Shape Dynamics

We don't want to only get solutions from CMC foliable space-times. Work directly with the Shape Dynamics equations of motion, tractable in the maximal slicing $\pi \approx 0$ case. (for open manifolds).

Shape Dynamics exact solutions for asymptotic flat boundary conditions

For conformally flat, static universe, $(g_{ab}(t), \pi^{ab}(t)) = (\Omega\delta_{ab}, 0)$:

- For empty Universe we get:
  - Solutions can be translated back into the usual Minkowski and Rindler space-times.
  - One for each type of unfixed (or "untraded") asymptotic time displacements: $N_o = \{1, x^a\}$
- For non-zero mass (as included in the asymptotic boundary conditions):
  - Solution translated back is isotropic black hole.

Work in progress: Isotropic black hole

Solution is: $N_o = \frac{1-m/(2r)}{1+m/(2r)}$ and $\Omega_o = 1 + \frac{m}{2r}$.

At $r = m/2$, $N_o = 0 = \det g_{\mu\nu}$.

- Equivalence between Shape Dynamics and GR breaks down.

Reconstructed 'spacetime' metric becomes:

$$ds^2 = -\left(\frac{1-m/(2r)}{1+m/(2r)}\right)^2 dt^2 - \left(1 + \frac{m}{2r}\right)^4 (dr^2 + r^2 d\Omega^2)$$

- No proper (vaccum) spacetime description valid for solution as a whole; only possible to consider $r > m/2$ and $r < m/2$.
- Infalling radial observer takes infinite proper time to reach $r = 0$. Raychaudhuri equation suffers discontinuity at $r = m/2$. Change of radial coord. $r \to m^2/(4r)$ leaves line element invariant.
- No infalling observer intersects a maximal slice more than once.

Collapse of the lapse, $N \to 0$, is a generic feature of maximal slicing of spacetimes describing black hole formation via gravitational collapse.

Conclusions

- ADM entangles evolution with constraints. Constraints quadratic in momenta.
  - Shape Dynamics disentangles dynamics from constraints; all constraints linear momenta, generate geometric transformations. Price: complicated form of the Hamiltonian H_{gl}.
- CMC gauge is in general only locally attainable and may be inhibited by global obstructions in spacetime.
- From the perspective of Shape Dynamics, there may be solutions for which the translation into a spacetime is obstructed, because the spacetime metric reconstructed from a perfectly regular Shape Dynamics solution may be degenerate.

Conclusions

- We have performed the asymptotic analysis for the usual flat asymptotic boundary conditions in Shape Dynamics.
  - We obtain the usual asymptotic Poincaré algebra.
  - Obtain the usual charges EXCEPT for an additional term for the energy, which makes it Weyl invariant.
- Explicit coincidence of ADM and Shape Dynamics for conditions assumed in locally asymptotically AdS. Needs further investigation.
- Fact that black hole formation via gravitational collapse involves a collapse of the lapse for maximal slicing doesn't say that Shape Dynamics doesn't have singularities of course. But if it does, the solutions won't be translatable to GR.

Finally, maybe we don't need to abandon GR, just Lorentz invariance.