Kavli IPMU focus week on gravity and Lorentz violations (2013)

Inflationary non-Gaussianities in the most general scalar-tensor theories

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Physical Review D84, 083504 (2011), arXiv:1107.3917
and arXiv:1301.5721

Inflationary models

Many inflationary models have been proposed so far.

- **Curvature inflation** (first model of inflation)

The higher-order curvature term leads to inflation.

Lagrangian: $f(R) = R + R^2/(6M^2)$ Starobinsky (1980)

- **"Old" inflation**

Inflation occurs due to the first-order phase transition of a vacuum.

Sato (1980), Kazanas (1980), Guth (1980)

- **Slow-roll inflation** Inflation is driven by the potential energy of a scalar field.

New, chaotic, power-law, hybrid, natural, extra-natural, eternal, D-term, F-term, brane, oscillating, tachyon, hill-top, KKLMMT, … (too many)

- **K-inflation** Inflation is driven by the kinetic energy of a scalar field.

Ghost condensate, DBI, Galileon,…

- **Inflation in extended theories of gravity.**

Brans-Dicke, Higgs, Horndeski's theory,….

There are also multi-field models.

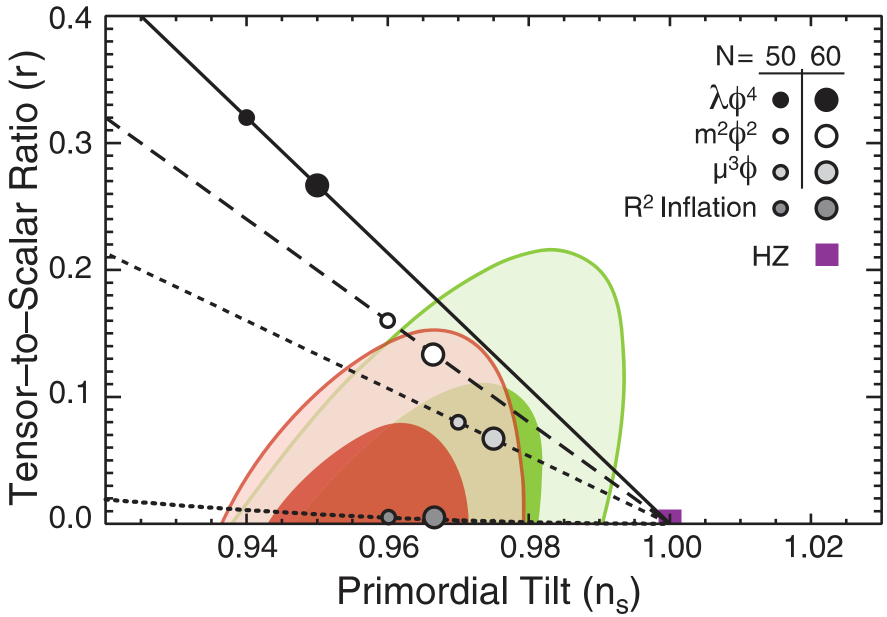
Inflationary observables

1. Scalar power spectrum $\mathcal{P}_{\mathcal{R}}(k) = \mathcal{P}_{\mathcal{R}}(k_0)\left(\frac{k}{k_0}\right)^{n_s-1}$ k_0 : pivot wavenumber

2. Tensor-to-scalar ratio $r = \frac{\mathcal{P}_h(k_0)}{\mathcal{P}_{\mathcal{R}}(k_0)}$ where $\mathcal{P}_h(k) = \mathcal{P}_h(k_0)\left(\frac{k}{k_0}\right)^{n_t}$

3. Non-linear parameter of scalar non-Gaussianities f_{NL}

WMAP9 bound combined with BAO and H_0
(without the running spectral index)



$n_s = 0.9608 \pm 0.0080$ (68 % CL)

$r < 0.13$ (95 % CL)

at $k_0 = 0.002\ \mathrm{Mpc}^{-1}$

The potential $V(\phi) = m^2\phi^2/2$
is in tension with the data.

In order to discriminate between a wide variety of inflationary models, we evaluate inflationary observables in the most general scalar-tensor theories with second-order equation of motion.

- This action was first derived by Horndeski in 1973.
- In 2011 Deffayet et al. derived the same action in a different form.
- The equivalence of two actions was shown by Kobayashi et al. in 2011.

This is the theory with one scalar degree of freedom, which covers a wide variety of single-field inflation models.

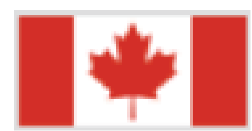
Horndeski's paper in 1973

International Journal of Theoretical Physics, Vol. 10, No. 6 (1974), pp. 363–384

Second-Order Scalar-Tensor Field Equations in a Four-Dimensional Space

Gregory Walter Horndeski

MathSciNet

Ph.D. University of Waterloo 1973 

Dissertation: *Invariant Variational Principles and Field Theories*

Advisor: David Lovelock

Horndeski's action

$$S = \int d^4x\sqrt{-g}\left[\frac{M_{\rm pl}^2}{2}R + P(\phi, X) - G_3(\phi, X)\,\Box\phi + \mathcal{L}_4 + \mathcal{L}_5\right].$$

$$\mathcal{L}_4 = G_4(\phi, X)\,R + G_{4,X}\left[(\Box\phi)^2 - (\nabla_\mu\nabla_\nu\phi)\,(\nabla^\mu\nabla^\nu\phi)\right]$$

$$\mathcal{L}_5 = G_5(\phi, X)\,G_{\mu\nu}\,(\nabla^\mu\nabla^\nu\phi) - \frac{1}{6}G_{5,X}[(\Box\phi)^3 - 3(\Box\phi)\,(\nabla_\mu\nabla_\nu\phi)\,(\nabla^\mu\nabla^\nu\phi) + 2(\nabla^\mu\nabla_\alpha\phi)\,(\nabla^\alpha\nabla_\beta\phi)\,(\nabla^\beta\nabla_\mu\phi)]$$

This action covers most of the dark energy models proposed in literature.

- Potential driven inflation: $P = X - V(\phi), \quad G_3 = 0, \quad G_4 = 0, \quad G_5 = 0$
- K-inflation: $P = P(\phi, X), \quad G_3 = 0, \quad G_4 = 0, \quad G_5 = 0$
- f(R) gravity and scalar-tensor gravity: $G_4 = F(\phi), \quad G_3 = 0, \quad G_5 = 0$
- Galileon: $P = -c_2 X, \quad G_3 = c_3 X/M^3, \quad G_4 = -c_4 X^2/M^6, \quad G_5 = 3c_5 X^2/M^9$
- Gauss-Bonnet coupling $\xi(\phi)\mathcal{G}$:

$$P = 8\xi^{(4)}(\phi)X^2(3 - \ln X), \quad G_3 = 4\xi^{(3)}(\phi)X(7 - 3\ln X)$$

$$G_4 = 4\xi^{(2)}(\phi)X(2 - \ln X), \quad G_5 = -4\xi^{(1)}(\phi)\ln X$$

Second-order action for curvature perturbations

We consider scalar metric perturbations $\alpha, \psi, \mathcal{R}$ with the ADM metric

$$ds^2 = -[(1+\alpha)^2 - a(t)^{-2}\, e^{-2\mathcal{R}}\,(\partial\psi)^2]\,dt^2 + 2\partial_i\psi\, dt\, dx^i + a(t)^2 e^{2\mathcal{R}} d\mathbf{x}^2$$

We choose the uniform field gauge: $\delta\phi = 0$

The second-order action for perturbations reduces to

$$S_2 = \int dt d^3x\, a^3 Q \left[\dot{\mathcal{R}}^2 - \frac{c_s^2}{a^2}\,(\partial\mathcal{R})^2\right]$$

$\Rightarrow$ $Q > 0$ and $c_s^2 > 0$ are required to avoid ghosts and Laplacian instabilities.

where $Q = \dfrac{w_1(4w_1w_3 + 9w_2^2)}{3w_2^2}\,, \qquad c_s^2 = \dfrac{3(2w_1^2 w_2 H - w_2^2 w_4 + 4w_1\dot{w}_1 w_2 - 2w_1^2\dot{w}_2)}{w_1(4w_1w_3 + 9w_2^2)}$

$$w_1 = M_{\rm pl}^2 F - 4XG_{4,X} - 2HX\dot{\phi}G_{5,X} + 2XG_{5,\phi}$$

$$\begin{aligned} w_2 = {} & 2M_{\rm pl}^2 HF - 2X\dot{\phi}G_{3,X} - 16H(XG_{4,X} + X^2G_{4,XX}) + 2\dot{\phi}(G_{4,\phi} + 2XG_{4,\phi X}) \\ & -2H^2\dot{\phi}(5XG_{5,X} + 2X^2G_{5,XX}) + 4HX(3G_{5,\phi} + 2XG_{5,\phi X}) \end{aligned}$$

$$\begin{aligned} w_3 = {} & -9M_{\rm pl}^2 H^2 F + 3(XP_{,X} + 2X^2P_{,XX}) + 18H\dot{\phi}(2XG_{3,X} + X^2G_{3,XX}) - 6X(G_{3,\phi} + XG_{3,\phi X}) \\ & +18H^2(7XG_{4,X} + 16X^2G_{4,XX} + 4X^3G_{4,XXX}) - 18H\dot{\phi}(G_{4,\phi} + 5XG_{4,\phi X} + 2X^2G_{4,\phi XX}) \\ & +6H^3\dot{\phi}(15XG_{5,X} + 13X^2G_{5,XX} + 2X^3G_{,5XXX}) - 18H^2X(6G_{5,\phi} + 9XG_{5,\phi X} + 2X^2G_{5,\phi XX}) \end{aligned}$$

$$w_4 = M_{\rm pl}^2 F - 2XG_{5,\phi} - 2XG_{5,X}\ddot{\phi}$$

Power spectrum of curvature perturbations

$$\mathcal{R}(\tau,\boldsymbol{x}) = \frac{1}{(2\pi)^3}\int d^3k\,\mathcal{R}(\tau,\boldsymbol{k})e^{i\boldsymbol{k}\cdot\boldsymbol{x}}\,, \qquad \mathcal{R}(\tau,\boldsymbol{k}) = u(\tau,\boldsymbol{k})a(\boldsymbol{k})+u^*(\tau,-\boldsymbol{k})a^\dagger(-\boldsymbol{k})$$

Introducing a rescaled field $v = zu$ with $z = a\sqrt{2Q}$, it follows that

$$v'' + \left(c_s^2k^2 - \frac{z''}{z}\right)v = 0$$

Under the slow-variation approximation we have

$$\frac{z''}{z} = 2(aH)^2\left[1+\mathcal{O}(\epsilon)\right], \qquad \epsilon = -\frac{\dot{H}}{H^2}$$

In the de Sitter limit the conformal time is $\tau = -1/(aH)$, in which case the solution is

$$u(\tau,k) = \frac{i\,H\,e^{-ic_s\tau}}{2(c_sk)^{3/2}\sqrt{Q}}(1+ic_sk\tau)$$

Using this solution in the limit $\tau \to 0$, the power spectrum of $\mathcal{R}$ is

$$\mathcal{P}_\mathcal{R} = \frac{H^2}{8\pi^2M_{\rm pl}^2\epsilon_sFc_s}$$

where $\epsilon_s \equiv \frac{Qc_s^2}{M_{\rm pl}^2F}$, $F = 1+\frac{2G_4}{M_{\rm pl}^2}$

The spectral index is

$$n_s - 1 \equiv \left.\frac{d\ln\mathcal{P}_\mathcal{R}}{d\ln k}\right|_{c_sk=aH} = -2\epsilon - \eta_{sF} - s$$

Kobayashi et al. (2011)
De Felice and S.T. (2011)

where $\eta_{sF} \equiv \frac{(\epsilon_sF)^\cdot}{H(\epsilon_sF)}$, $s \equiv \frac{\dot{c}_s}{Hc_s}$

Tensor power spectrum

Tensor perturbations: $h_{ij} = h_+ e_{ij}^+ + h_\times e_{ij}^\times$

The second-order action is

$$S_t = \sum_{\lambda=+,\times} \int dt d^3x a^3 Q_t \left[\dot{h}_\lambda^2 - \frac{c_t^2}{a^2} (\partial h_\lambda)^2 \right]$$

where $Q_t = \frac{w_1}{4}, \qquad c_t^2 = \frac{w_4}{w_1}$

The tensor power spectrum is $\mathcal{P}_t = \frac{H^2}{2\pi^2 Q_t c_t^3} \simeq \frac{2H^2}{\pi^2 M_{\rm pl}^2 F}$

The tensor-to-scalar ratio is $r = \frac{\mathcal{P}_t}{\mathcal{P}_\mathcal{R}} \simeq 16 c_s \epsilon_s$

These results can be readily used for concrete models of inflation.

e.g., $P = X - V(\phi)$

$$\Rightarrow \quad n_s = 1 - 6\epsilon_V + 2\eta_V, \qquad r = 16\epsilon_V$$

where $\epsilon_V = \frac{M_{\rm pl}^2}{2} \left(\frac{V_{,\phi}}{V} \right)^2, \quad \eta_V = \frac{M_{\rm pl}^2 V_{,\phi\phi}}{V}$

These observables depend on the slope of the potential.

Observational bounds

If PLANCK places the bound like $r < 0.01$, many slow-roll inflation models can be ruled out.

Lyth bound

$$\frac{\Delta\phi}{M_{\rm pl}} \gtrsim \left(\frac{r}{0.01}\right)^{1/2}$$

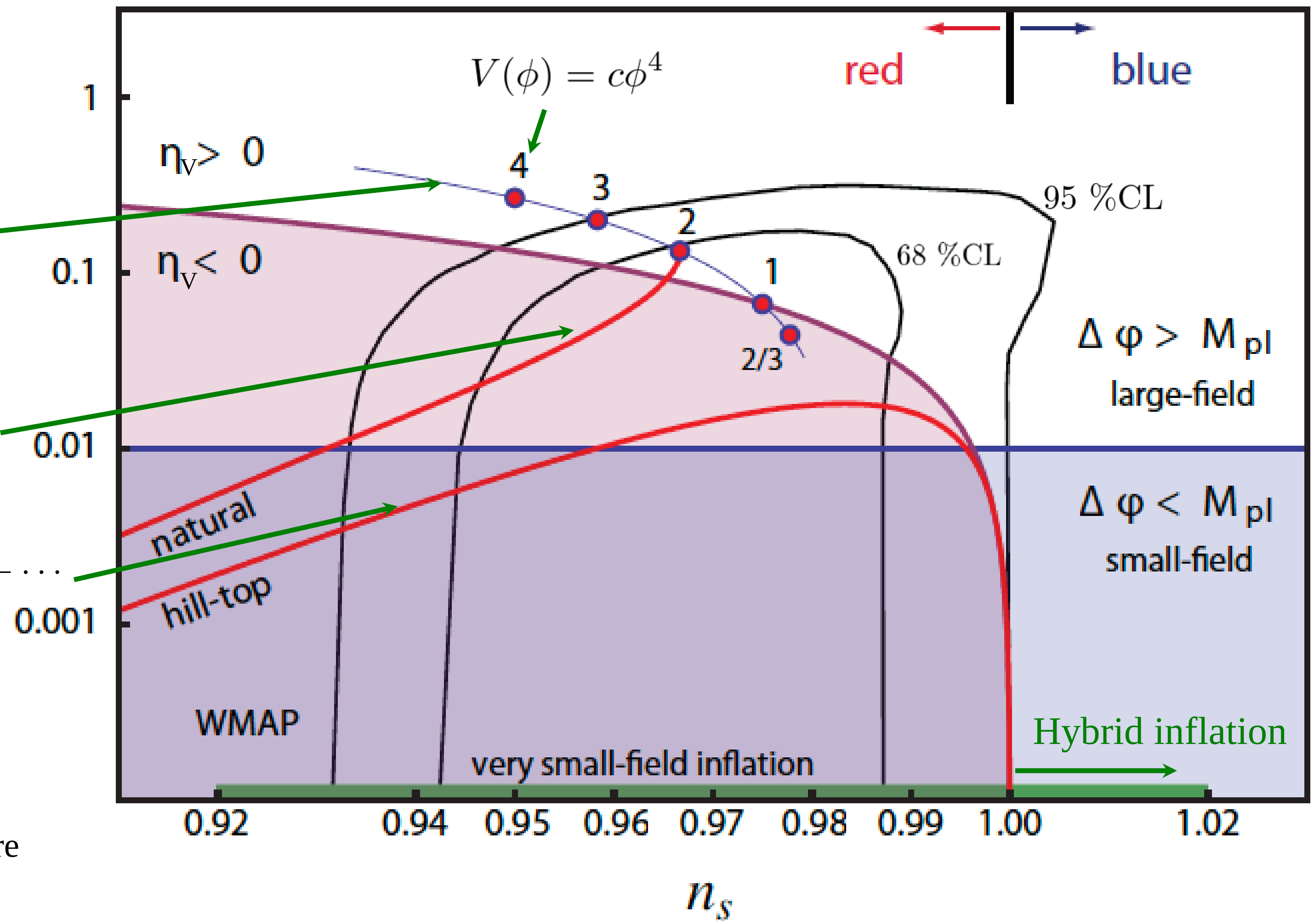
Chaotic inflation

$$V(\phi) = c\phi^p$$

Natural inflation

$$V(\phi) = V_0[1 - \cos(\phi/f)]$$

$$V(\phi) = V_0[1 - (\phi/M)^2] + \cdots$$



$$\eta_V = \frac{M_{\rm pl}^2 V_{,\phi\phi}}{V}$$

characterizes the curvature of the potential.

Scalar non-Gaussianities

The information of scalar non-Gaussianities is useful to place further constraints on inflationary models.

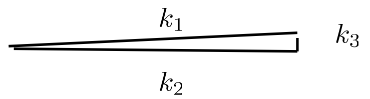
Three-point correlation function of curvature perturbations is

$$\langle \mathcal{R}(\mathbf{k}_1)\mathcal{R}(\mathbf{k}_2)\mathcal{R}(\mathbf{k}_3)\rangle = (2\pi)^7 \delta^{(3)}(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3)(\mathcal{P}_{\mathcal{R}})^2 \frac{\mathcal{A}_{\mathcal{R}}}{\prod_{i=1}^{3} k_i^3}$$

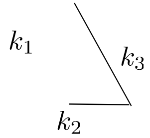
The non-linear parameter is defined as $f_{\rm NL} = \frac{10}{3}\frac{\mathcal{A}_{\mathcal{R}}}{\sum_{i=1}^{3} k_i^3}$

There are several different shapes of non-Gaussianities.

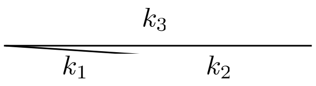
(i) Squeezed limit (local type) $k_3 \to 0, k_1 \sim k_2$



(ii) Equilateral type $k_1 = k_2 = k_3$



(iii) Enfolded type $k_3 = k_1 + k_2$



WMAP9 constraints on the nonlinear parameter

$$\begin{aligned}
f_{\rm NL}^{\rm local} &= 37.2 \pm 19.9 \quad (68\,\%\,{\rm CL})\,, & f_{\rm NL}^{\rm local} &= 37 \pm 40 \quad (95\,\%\,{\rm CL})\,,\\
f_{\rm NL}^{\rm equil} &= 51 \pm 136 \quad (68\,\%\,{\rm CL})\,, & f_{\rm NL}^{\rm equil} &= 51 \pm 272 \quad (95\,\%\,{\rm CL})\,,\\
f_{\rm NL}^{\rm ortho} &= -245 \pm 100 \quad (68\,\%\,{\rm CL})\,, & f_{\rm NL}^{\rm ortho} &= -245 \pm 200 \quad (95\,\%\,{\rm CL})\,.
\end{aligned}$$

The 'orthogonal' shape is orthogonal to the equilateral template (explained later).

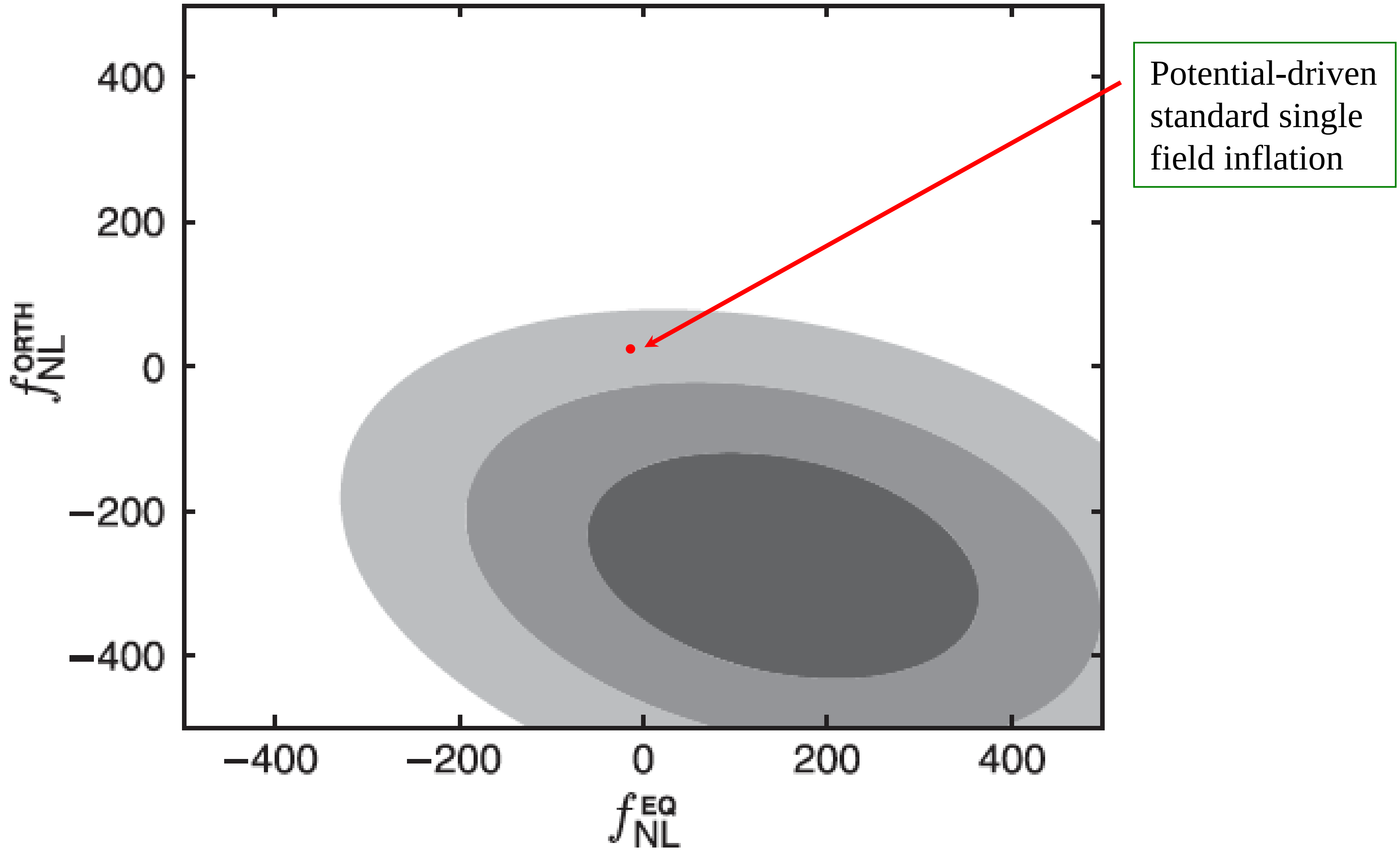
Using the relation $f_{\rm NL}^{\rm enfold} = (f_{\rm NL}^{\rm equil} - f_{\rm NL}^{\rm ortho})/2$, the enfolded nonlinear parameter is constrained to be

$$f_{\rm NL}^{\rm enfold} = 148 \pm 118 \quad (68\,\%\,{\rm CL})\,, \qquad f_{\rm NL}^{\rm enfold} = 148 \pm 236 \quad (95\,\%\,{\rm CL})$$

For the local, orthogonal, and enfolded shapes the model with purely Gaussian perturbations ($f_{\rm NL} = 0$) is outside the 68 % observational contour, but, apart from the orthogonal case, it is still consistent with the WMAP constraints at 95 % CL.

We evaluate the nonlinear parameter in the Horndeski's theory which can be used for any shape of non-Gaussianities.

WMAP9 2-dimensional constraints on $f_{\mathrm{N}}^{\mathrm{equil}}$ and $f_{\mathrm{N}}^{\mathrm{ortho}}$



The third-order action for scalar perturbations

$$\mathcal{S}_3 = \int dt\, d^3x \Big\{ a^3 \mathcal{C}_1 M_{\rm pl}^2 \mathcal{R}\dot{\mathcal{R}}^2 + a\,\mathcal{C}_2 M_{\rm pl}^2 \mathcal{R}(\partial\mathcal{R})^2 + a^3\mathcal{C}_3 M_{\rm pl}\dot{\mathcal{R}}^3 + a^3\mathcal{C}_4\dot{\mathcal{R}}(\partial_i\mathcal{R})(\partial_i\mathcal{X}) + a^3(\mathcal{C}_5/M_{\rm pl}^2)\partial^2\mathcal{R}(\partial\mathcal{X})^2$$
$$+ a\mathcal{C}_6\dot{\mathcal{R}}^2\partial^2\mathcal{R} + \mathcal{C}_7\left[\partial^2\mathcal{R}(\partial\mathcal{R})^2 - \mathcal{R}\partial_i\partial_j(\partial_i\mathcal{R})(\partial_j\mathcal{R})\right]/a + a(\mathcal{C}_8/M_{\rm pl})\left[\partial^2\mathcal{R}\partial_i\mathcal{R}\partial_i\mathcal{X} - \mathcal{R}\partial_i\partial_j(\partial_i\mathcal{R})(\partial_j\mathcal{X})\right]$$
$$+ \mathcal{F}_1 \frac{\delta\mathcal{L}_2}{\delta\mathcal{R}}\bigg|_1 \Big\}$$

where $\partial^2\mathcal{X} = Q\dot{\mathcal{R}}$ and $\mathcal{C}_i$ are coefficients which depend on the background.

$\mathcal{F}_1$ includes the time and spatial derivatives of $\mathcal{R}$ and $\mathcal{X}$ alone, and

$$\frac{\delta\mathcal{L}_2}{\delta\mathcal{R}}\bigg|_1 \equiv -2\left[\frac{d}{dt}(a^3 Q\dot{\mathcal{R}}) - aQc_s^2\partial^2\mathcal{R}\right]$$

The vacuum expectation value of $\mathcal{R}$ for the three-point operator in the asymptotic future ($\tau \to 0$) is

$$\langle \mathcal{R}(\mathbf{k}_1)\mathcal{R}(\mathbf{k}_2)\mathcal{R}(\mathbf{k}_3)\rangle = -i\int_{-\infty}^{0} d\tau\, a\,\langle 0|\left[\mathcal{R}(0,\mathbf{k}_1)\mathcal{R}(0,\mathbf{k}_2)\mathcal{R}(0,\mathbf{k}_3), \mathcal{H}_{\rm int}(\tau)\right]|0\rangle$$

where $\mathcal{H}_{\rm int} = -\mathcal{L}_3$ and $\mathcal{S}_3 = \int dt\,\mathcal{L}_3$.

We write the three-point correlation function in the form

$$\langle \mathcal{R}(\mathbf{k}_1)\mathcal{R}(\mathbf{k}_2)\mathcal{R}(\mathbf{k}_3)\rangle = (2\pi)^3\delta^{(3)}(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3)(\mathcal{P}_{\mathcal{R}})^2\mathcal{F}_{\mathcal{R}}(k_1,k_2,k_3)$$

where

$$\mathcal{F}_{\mathcal{R}}(k_1,k_2,k_3) = \frac{(2\pi)^4}{\prod_{i=1}^3 k_i^3}\mathcal{A}_{\mathcal{R}}(k_1,k_2,k_3)$$

The bispectrum $\mathcal{R}$ can be evaluated under the slow-variation approximation.

The leading-order bispectrum

Using the mode function on the de Sitter background ($a = -1/(H\tau)$) and assuming that $\mathcal{C}_i$'s are positive, the bispectrum is

$$\mathcal{A}_\mathcal{R} \supset \frac{c_s^2}{4\epsilon_s F}\mathcal{C}_1 S_1 + \frac{1}{4\epsilon_s F}\mathcal{C}_2 S_2 + \frac{3c_s^2}{2\epsilon_s F}\frac{H}{M_{\rm pl}}\mathcal{C}_3 S_3 + \frac{1}{8}\mathcal{C}_4 S_4 + \frac{\epsilon_s F}{4c_s^2}\mathcal{C}_5 S_5 + \frac{3}{\epsilon_s F}\left(\frac{H}{M_{\rm pl}}\right)^2 \mathcal{C}_6 S_6 + \frac{1}{2\epsilon_s F c_s^2}\left(\frac{H}{M_{\rm pl}}\right)^2 \mathcal{C}_7 S_7 + \frac{1}{8c_s^2}\frac{H}{M_{\rm pl}}\mathcal{C}_8 S_8\,,$$

Gao and Steer (2011), De Felice and S.T. (2011)

where the shape functions are

$$S_1 = \frac{2}{K}\sum_{i>j}k_i^2k_j^2 - \frac{1}{K^2}\sum_{i\neq j}k_i^2k_j^3, \qquad S_2 = \frac{1}{2}\sum_i k_i^3 + \frac{2}{K}\sum_{i>j}k_i^2k_j^2 - \frac{1}{K^2}\sum_{i\neq j}k_i^2k_j^3, \qquad S_3 = \frac{(k_1k_2k_3)^2}{K^3},$$

$$S_4 = \sum_i k_i^3 - \frac{1}{2}\sum_{i\neq j}k_ik_j^2 - \frac{2}{K^2}\sum_{i\neq j}k_i^2k_j^3\,, \qquad S_5 = \frac{1}{K^2}\left[\sum_i k_i^5 + \frac{1}{2}\sum_{i\neq j}k_ik_j^4 - \frac{3}{2}\sum_{i\neq j}k_i^2k_j^3 - k_1k_2k_3\sum_{i>j}k_ik_j\right],$$

Present in k-inflation

$$S_6 = S_3, \qquad S_7 = \frac{1}{K}\left(1 + \frac{1}{K^2}\sum_{i>j}k_ik_j + \frac{3k_1k_2k_3}{K^3}\right)\left[\frac{3}{4}\sum_i k_i^4 - \frac{3}{2}\sum_{i>j}k_i^2k_j^2\right],$$

$$S_8 = \frac{1}{K^2}\left[\frac{3}{2}k_1k_2k_3\sum_i k_i^2 - \frac{5}{2}k_1k_2k_3K^2 - 6\sum_{i\neq j}k_i^2k_j^3 - \sum_i k_i^5 + \frac{7}{2}K\sum_i k_i^4\right]$$

These appear in the Horndeski's theory.

and $K = k_1 + k_2 + k_3$.

S_7 and S_8 can be expressed as $\quad S_7 = -\frac{3}{2}(3S_1 - S_2) + 18S_3\,, \qquad S_8 = 3S_1 - S_2 + 3S_4.$

Simplification of the bispectrum

The coefficients are the order $\mathcal{C}_i = \mathcal{O}(\epsilon)$.

$$\mathcal{A}_{\mathcal{R}} \supset \frac{c_s^2}{4\epsilon_s F}\mathcal{C}_1 S_1 + \frac{1}{4\epsilon_s F}\mathcal{C}_2 S_2 + \frac{3c_s^2}{2\epsilon_s F}\frac{H}{M_{\rm pl}}\mathcal{C}_3 S_3 + \frac{1}{8}\mathcal{C}_4 S_4 + \underline{\frac{\epsilon_s F}{4c_s^2}\mathcal{C}_5 S_5 + \frac{3}{\epsilon_s F}\left(\frac{H}{M_{\rm pl}}\right)^2 \mathcal{C}_6 S_6 + \frac{1}{2\epsilon_s F c_s^2}\left(\frac{H}{M_{\rm pl}}\right)^2 \mathcal{C}_7 S_7 + \frac{1}{8c_s^2}\frac{H}{M_{\rm pl}}\mathcal{C}_8 S_8}\,,$$

$\mathcal{O}(\epsilon^2)$. Neglected.

$$\mathcal{A}_{\mathcal{R}} \supset \underbrace{\frac{c_s^2}{4\epsilon_s F}\tilde{\mathcal{C}}_1 S_1 + \frac{1}{4\epsilon_s F}\tilde{\mathcal{C}}_2 S_2 + \frac{3c_s^2}{2\epsilon_s F}\frac{H}{M_{\rm pl}}\tilde{\mathcal{C}}_3 S_3}_{\mathcal{O}(\epsilon^0)} + \underbrace{\frac{1}{8}\tilde{\mathcal{C}}_4 S_4}_{\mathcal{O}(\epsilon)}$$

where $\tilde{\mathcal{C}}_1 = \mathcal{C}_1 - \frac{3H^2}{2c_s^4 M_{\rm pl}^2}(6 + 2\epsilon + 7\eta_{sF} - 5\eta_7)\mathcal{C}_7 + \frac{3H\epsilon_s F}{2c_s^4 M_{\rm pl}}\mathcal{C}_8$ etc

$\boxed{\text{If the leading-order bispectrum vanishes (which happens in the local limit), we need to take into account the } \mathcal{O}(\epsilon) \text{ corrections to } \tilde{\mathcal{C}}_1,\ \tilde{\mathcal{C}}_2,\ \tilde{\mathcal{C}}_3.}$

Slow-variation corrections

(i) The variation of the coefficients $\mathcal{C}_i$

$$\boxed{\tilde{\mathcal{C}}_i(\tau) = \tilde{\mathcal{C}}_i(\tau_K) - \frac{d\tilde{\mathcal{C}}_i}{dt}\frac{1}{H_K}\ln\frac{\tau}{\tau_K} + \mathcal{O}(\epsilon^2\tilde{\mathcal{C}}_i)}$$

where $\tau_K = -1/(Kc_{sK})$

(ii) The correction to the scale factor

$$\boxed{a = -\frac{1}{H_K\tau} - \frac{\epsilon}{H_K\tau} + \frac{\epsilon}{H_K\tau}\ln(\tau/\tau_K) + \mathcal{O}(\epsilon^2)}$$

(iii) The correction to the mode function

$$\boxed{u(y) = -\frac{\sqrt{\pi}}{2\sqrt{2}}\frac{1}{\sqrt{\epsilon_s F c_s}}\frac{H}{M_{\rm pl}}\frac{y^{3/2}}{k^{3/2}}\left(1+\frac{1}{2}\epsilon+\frac{1}{2}s\right)e^{i\frac{\pi}{2}(\epsilon+\frac{1}{2}\eta_{sF})}H_\nu^{(1)}[(1+\epsilon+s)y]}$$

where $y = c_s k/(aH)$, $\nu = 3/2 + \epsilon + \eta_{sF}/2 + s/2$

The corrections to the bispectrum can be written as

$$\Delta\mathcal{A}_{\mathcal{R}}^{(1)} = \left(\frac{c_s^2}{4\epsilon_s F}\tilde{\mathcal{C}}_1^{\rm lead}\right)_K \delta Q_1\,, \qquad \Delta\mathcal{A}_{\mathcal{R}}^{(2)} = -\left(\frac{1}{4\epsilon_s F}\tilde{\mathcal{C}}_2^{\rm lead}\right)_K \delta Q_2\,, \qquad \Delta\mathcal{A}_{\mathcal{R}}^{(3)} = \left(\frac{3c_s^2 H}{4\epsilon_s F M_{\rm pl}}\tilde{\mathcal{C}}_3^{\rm lead}\right)_K \delta Q_3$$

δQ_i $(i=1,2,3)$ are the terms of the order of $\mathcal{O}(\epsilon)$.

Full bispectrum

De Felice and S.T. (2013)

$$\mathcal{A}_\mathcal{R} = \mathcal{A}_\mathcal{R}^{\rm lead} + \mathcal{A}_\mathcal{R}^{\rm corre}$$

where

$$\mathcal{A}_\mathcal{R}^{\rm lead} = \left[\frac{1}{4}\left(1-\frac{1}{c_s^2}\right)+\frac{1}{2c_s^2}\frac{\delta\mathcal{C}_7}{\epsilon_s}\right](3S_1-S_2)+\left[\frac{3}{2}\left(\frac{1}{c_s^2}-1\right)-\frac{3\lambda}{\Sigma}+\frac{6\delta\mathcal{C}_6}{\epsilon_s}-\frac{6}{c_s^2}\frac{\delta\mathcal{C}_7}{\epsilon_s}\right]S_3\,,$$

$$\begin{aligned}\mathcal{A}_\mathcal{R}^{\rm corre} =& \frac{1}{4c_s^2}\left[\delta\mathcal{C}_1+(2\epsilon+7\eta_{sF}-5\eta_7)\frac{\delta\mathcal{C}_7}{\epsilon_s}+3\delta\mathcal{C}_8\right]S_1+\frac{1}{4c_s^2}\left[\delta\mathcal{C}_2+(2\epsilon-\eta_{sF}-\eta_7+4s)\frac{\delta\mathcal{C}_7}{\epsilon_s}-\delta\mathcal{C}_8\right]S_2\\ &+\left[\frac{3}{2}\delta\mathcal{C}_3+(3\eta_{sF}-\eta_6-4s)\frac{\delta\mathcal{C}_6}{\epsilon_s}-\frac{1}{c_s^2}(3\eta_{sF}-2\eta_7-s)\frac{\delta\mathcal{C}_7}{\epsilon_s}\right]S_3-\frac{1}{4c_s^2}(\epsilon_s-3\delta\mathcal{C}_8)S_4\\ &-\frac{1}{4}\left(\frac{1}{c_s^2}-1-\frac{2}{c_s^2}\frac{\delta\mathcal{C}_7}{\epsilon_s}\right)(3\delta Q_1+\delta Q_2)+\frac{3}{4}\left(\frac{1}{c_s^2}-1-\frac{2\lambda}{\Sigma}+4\frac{\delta\mathcal{C}_6}{\epsilon_s}-\frac{4}{c_s^2}\frac{\delta\mathcal{C}_7}{\epsilon_s}\right)\delta Q_3\end{aligned}$$

where $\delta\mathcal{C}_i$'s are the order of $\mathcal{O}(\epsilon)$ and

$$\Sigma = \frac{w_1(4w_1w_3+9w_2^2)}{12M_{\rm pl}^4}\,,$$

$$\begin{aligned}\lambda =& \frac{F^2}{3}[3X^2P_{,XX}+2X^3P_{,XXX}+3H\dot\phi(XG_{3,X}+5X^2G_{3,XX}+2X^3G_{3,XXX})-2(2X^2G_{3,\phi X}+X^3G_{3,\phi XX})\\ &+6H^2(9X^2G_{4,XX}+16X^3G_{4,XXX}+4X^4G_{4,XXXX})-3H\dot\phi(3XG_{4\phi,X}+12X^2G_{4,\phi XX}+4X^3G_{4,\phi XXX})\\ &+H^3\dot\phi(3XG_{5,X}+27X^2G_{5,XX}+24X^3G_{5,XXX}+4X^4G_{5,XXXX})\\ &-6H^2(6X^2G_{5,\phi X}+9X^3G_{5,\phi XX}+2X^4G_{5,\phi XXX})\end{aligned}$$

Local non-Gaussianities

● **Squeezed limit:** $k_3 \to 0, \quad k_2 \to k_1 \equiv k$

$$S_2 = 3k^3/2 = 3S_1, \; S_3 = S_4 = 0 \quad \Longrightarrow \quad \boxed{\mathcal{A}_{\mathcal{R}}^{\text{lead}} = 0}$$

$$\mathcal{A}_{\mathcal{R}}^{\text{corre}} = \frac{k^3}{4}(2\epsilon + \eta_{sF} + s) \quad \Longrightarrow \quad \boxed{f_{\text{NL}}^{\text{local}} = \frac{5}{12}(1 - n_s) \ll 1}$$

Same as that derived by Maldacena in standard slow-roll inflation.

The small local non-Gaussianity in the squeezed limit is a generic feature in the single-field slow-variation inflation.

● **Not-so squeezed case:** $r_3 = k_3/k_1 \neq 0$, but $r_3 \ll 1$

The leading-order bispectrum has the dependence $|f_{\text{NL}}^{\text{lead}}| \approx r_3^2/c_s^2$

This dominates over the correction for

$$r_3 > c_s\sqrt{1 - n_s} \quad \Longrightarrow \quad r_3 > 0.2c_s \text{ for } n_s = 0.96$$

For the models with $c_s^2 \ll 1$ there is the growth of $|f_{\text{NL}}|$ with the increase of r_3 from 0.

Equilateral and enfolded non-Gaussianities

● Equilateral non-Gaussianity $k_1 = k_2 = k_3$

$\lambda_{3X} = X G_{3,XX}/G_{3,X}$
etc

$$\boxed{\begin{aligned} f_{\rm NL}^{\rm equil,lead} = {} & \frac{85}{324}\left(1-\frac{1}{c_s^2}\right) - \frac{10}{81}\frac{\lambda}{\Sigma} + \frac{20}{81\epsilon_s}\left[(1+\lambda_{3X})\delta_{G3X} + 4(3+2\lambda_{4X})\delta_{G4XX} + \delta_{G5X} + (5+2\lambda_{5X})\delta_{G5XX}\right] \\ & + \frac{65}{162 c_s^2 \epsilon_s}\left(\delta_{G3X} + 6\delta_{G4XX} + \delta_{G5X} + \delta_{G5XX}\right) \end{aligned}}$$

$\Rightarrow$ $|f_{\rm NL}^{\rm equil,lead}| \gg 1$ for $c_s^2 \ll 1$

In standard slow-roll inflation with $G_3 = G_4 = G_5 = 0$ we have

$f_{\rm NL}^{\rm equil,lead} = 0$ and $f_{\rm NL}^{\rm equil,corre} = 55\epsilon_s/36 + 5\eta_s/12.$ $\Rightarrow$ small

● Enfolded non-Gaussianity $k_3 \to k, k_1 \to k_2 = k/2$

$$\boxed{f_{\rm NL}^{\rm enfold,lead} = \frac{1}{32}\left(1-\frac{1}{c_s^2}\right) - \frac{1}{16}\frac{\lambda}{\Sigma} + \frac{1}{8\epsilon_s}\left[(1+\lambda_{3X})\delta_{G3X} + 4(3+2\lambda_{4X})\delta_{G4XX} + \delta_{G5X} + (5+2\lambda_{5X})\delta_{G5XX}\right]}$$

$\Rightarrow$ $|f_{\rm NL}^{\rm enfold,lead}| \gg 1$ for $c_s^2 \ll 1$

In standard slow-roll inflation with $G_3 = G_4 = G_5 = 0$ we have

$f_{\rm NL}^{\rm enfold,lead} = 0$ and $f_{\rm NL}^{\rm enfold,corre} = 7\epsilon_s/8 + 5\eta_s/12.$ $\Rightarrow$ small

Shapes of non-Gaussianities

The CMB data analysis of non-Gaussianities have been carried out by using the factorizable shape functions:

$$\mathcal{F}_{\mathcal{R}} = (2\pi)^4 \frac{\mathcal{A}_{\mathcal{R}}}{\prod_{i=1}^{3} k_i^3}$$

(i) Local template

$$\mathcal{F}_{\mathcal{R}}^{\rm local}(k_1,k_2,k_3) = (2\pi)^4 \left(\frac{3}{10} f_{\rm NL}^{\rm local}\right)\left(\frac{1}{k_1^3 k_2^3} + \frac{1}{k_2^3 k_3^3} + \frac{1}{k_3^3 k_1^3}\right)$$

(ii) Equilateral template

$$\mathcal{F}_{\mathcal{R}}^{\rm equil}(k_1,k_2,k_3) = (2\pi)^4 \left(\frac{9}{10} f_{\rm NL}^{\rm equil}\right)\left[-\frac{1}{k_1^3 k_2^3} - \frac{1}{k_2^3 k_3^3} - \frac{1}{k_3^3 k_1^3} - \frac{2}{k_1^2 k_2^2 k_3^2} + \left(\frac{1}{k_1 k_2^2 k_3^3} + 5\ \text{perm.}\right)\right]$$

(iii) Orthogonal template

Orthogonal to the equilateral one (the correlation with the equilateral template is small).

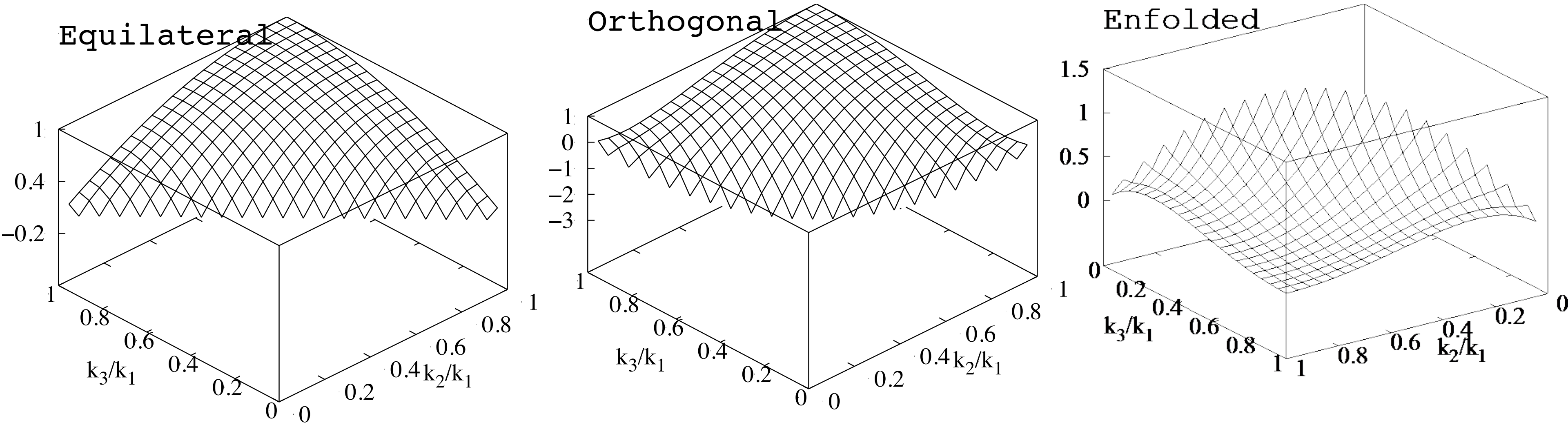
$$\mathcal{F}_{\mathcal{R}}^{\rm ortho}(k_1,k_2,k_3) = (2\pi)^4 \left(\frac{9}{10} f_{\rm NL}^{\rm ortho}\right)\left[-\frac{3}{k_1^3 k_2^3} - \frac{3}{k_2^3 k_3^3} - \frac{3}{k_3^3 k_1^3} - \frac{8}{k_1^2 k_2^2 k_3^2} + \left(\frac{3}{k_1 k_2^2 k_3^3} + 5\ \text{perm.}\right)\right]$$

(iv) Enfolded template

$$\mathcal{F}_{\mathcal{R}}^{\rm enfold} = (\mathcal{F}_{\mathcal{R}}^{\rm equil} - \mathcal{F}_{\mathcal{R}}^{\rm ortho})/2$$

$$\mathcal{F}_{\mathcal{R}}^{\rm enfold}(k_1,k_2,k_3) = (2\pi)^4 \left(\frac{9}{10} f_{\rm NL}^{\rm enf}\right)\left[\frac{1}{k_1^3 k_2^3} + \frac{1}{k_2^3 k_3^3} + \frac{1}{k_3^3 k_1^3} + \frac{3}{k_1^2 k_2^2 k_3^2} - \left(\frac{1}{k_1 k_2^2 k_3^3} + 5\ \text{perm.}\right)\right]$$

Templates



The correlation between two different templates

$$\boxed{C(\mathcal{F}_{\mathcal{R}}^{(i)}, \mathcal{F}_{\mathcal{R}}^{(j)}) = \frac{\mathcal{I}(\mathcal{F}_{\mathcal{R}}^{(i)}, \mathcal{F}_{\mathcal{R}}^{(j)})}{\sqrt{\mathcal{I}(\mathcal{F}_{\mathcal{R}}^{(i)}, \mathcal{F}_{\mathcal{R}}^{(i)})\,\mathcal{I}(\mathcal{F}_{\mathcal{R}}^{(j)}, \mathcal{F}_{\mathcal{R}}^{(j)})}}}$$

where $\mathcal{I}(\mathcal{F}_{\mathcal{R}}^{(i)}, \mathcal{F}_{\mathcal{R}}^{(j)}) = \int d\mathcal{V}_k\, \mathcal{F}_{\mathcal{R}}^{(i)}(k_1, k_2, k_3)\mathcal{F}_{\mathcal{R}}^{(j)}(k_1, k_2, k_3)\frac{(k_1k_2k_3)^4}{(k_1+k_2+k_3)^3}$

e.g., $C(\mathcal{F}_{\mathcal{R}}^{\rm equil}, \mathcal{F}_{\mathcal{R}}^{\rm ortho}) = 0.025$ $\quad C(\mathcal{F}_{\mathcal{R}}^{\rm equil}, \mathcal{F}_{\mathcal{R}}^{\rm enfold}) = 0.512$

Horndeski's theory

The leading-order bispectrum is

$$\mathcal{A}_{\mathcal{R}}^{\text{lead}} = \left[\frac{1}{4}\left(1-\frac{1}{c_s^2}\right)+\frac{1}{2c_s^2}\frac{\delta\mathcal{C}_7}{\epsilon_s}\right]\underline{(3S_1-S_2)}+\left[\frac{3}{2}\left(\frac{1}{c_s^2}-1\right)-\frac{3\lambda}{\Sigma}+\frac{6\delta\mathcal{C}_6}{\epsilon_s}-\frac{6}{c_s^2}\frac{\delta\mathcal{C}_7}{\epsilon_s}\right]\underline{S_3}$$

The correlation with the equilateral template is large, but there is also the contribution from the orthogonal one.

The shape functions $3S_1 - S_2$ and S_3 are related with S_7, as

$$\boxed{S_7 = -\frac{3}{2}(3S_1 - S_2) + 18S_3}$$

The correlation with the equilateral template is

$C = 0.999892$ (very close to 1)

We can use the following equilateral and orthogonal bases to rewrite the bispectrum.

$$\boxed{S_7^{\text{equil}} = -\frac{12}{13}S_7}$$

The shape fucntion $\mathcal{F}_7^{\text{equil}}$ is normalized to be 1 at $k_1 = k_2 = k_3$.

$$\boxed{S_7^{\text{ortho}} = \frac{12}{14-13\beta}(\beta S_7 + 3S_1 - S_2)}$$

$$C(S_7^{\text{equil}}, S_7^{\text{ortho}}) = 0$$

where $\beta = \frac{16}{3}\frac{248041 - 25200\pi^2}{1986713 - 201600\pi^2}$

Leading-order bispectrum

$$\boxed{\mathcal{A}_{\mathcal{R}}^{\rm lead} = c_1 S_7^{\rm equil} + c_2 S_7^{\rm ortho}}$$

where

$$\begin{aligned} c_1 &= \frac{13}{12}\left[\frac{1}{24}\left(1-\frac{1}{c_s^2}\right)(2+3\beta)+\frac{\lambda}{12\Sigma}(2-3\beta)-\frac{\delta\mathcal{C}_6}{6\epsilon_s}(2-3\beta)+\frac{\delta\mathcal{C}_7}{3\epsilon_s c_s^2}\right], \\ c_2 &= \frac{14-13\beta}{12}\left[\frac{1}{8}\left(1-\frac{1}{c_s^2}\right)-\frac{\lambda}{4\Sigma}+\frac{\delta\mathcal{C}_6}{2\epsilon_s}\right] \qquad (\beta \simeq 1.2) \end{aligned}$$

The coefficients c_1 and c_2 characterise the contributions of equilateral and orthogonal shapes.

This depends on the models of inflation. For the models characterized by

$$\delta\mathcal{C}_7 = \delta_{G3X} + 6\delta_{G4XX} + \delta_{G5X} + \delta_{G5XX} \neq 0,$$

the orthogonal contrubition can be important (where $\delta_{G3X} = \frac{G_{3,X}\dot{\phi}X}{M_{\rm pl}^2 HF}$ etc).

Power-law k-inflation

Power-law k-inflation ($a \propto t^{1/\gamma}$ with $\gamma \ll 1$) can be realized for

$$\boxed{P(\phi, X) = K(\phi)(-X + X^2)\,, \quad G_3 = 0\,, \quad G_4 = 0\,, \quad G_5 = 0} \qquad \boxed{K(\phi) \propto \phi^{-2}}$$

In this case one has

$$X = \frac{3-\gamma}{3(2-\gamma)}\,, \qquad c_s^2 = \frac{\gamma}{3(4-\gamma)} \ll 1$$

In the limit that $c_s^2 \ll 1$ we have

$$\boxed{\mathcal{A}_{\mathcal{R}}^{\rm lead} \simeq (-0.252/c_s^2) S_7^{\rm equil} + (0.016/c_s^2) S_7^{\rm ortho}}$$

In this case the equilateral shape dominates.

$$f_{\rm NL}^{\rm equil,lead} \simeq -85/(324 c_s^2) \quad \text{and} \quad f_{\rm NL}^{\rm enfold,lead} \simeq -1/(32 c_s^2)$$

The WMAP9 bound gives $c_s^2 > 1.2 \times 10^{-3}$ (95 % CL)

Galileon inflation

Model: $P(X) = -X + X^2/(2M^4)$ with the Galileon terms

$$\boxed{G_3(X) = \mu X/M^4, \qquad G_4(X) = \mu X^2/M^7, \qquad G_5(X) = \mu X^2/M^{10}}$$

In the limit that $c_s^2 \ll 1$ the bispectra are

(i) $G_3(X) = \mu X/M^4$

$$\mathcal{A}_{\mathcal{R}}^{\rm lead} \simeq (0.109/c_s^2) S_7^{\rm equil} + (0.016/c_s^2) S_7^{\rm ortho}$$

$$C^{\rm equil} = 0.972, \quad C^{\rm ortho} = 0.240 \quad \text{for } c_s^2 = 0.01$$

(ii) $G_4(X) = \mu X^2/M^7$

$$\mathcal{A}_{\mathcal{R}}^{\rm lead} \simeq (0.018/c_s^2) S_7^{\rm equil} + (0.016/c_s^2) S_7^{\rm ortho}$$

The orthogonal contribution is comparable to the equilateral one.

$$C^{\rm equil} = 0.684, \quad C^{\rm ortho} = 0.680 \quad \text{for } c_s^2 = 0.01$$

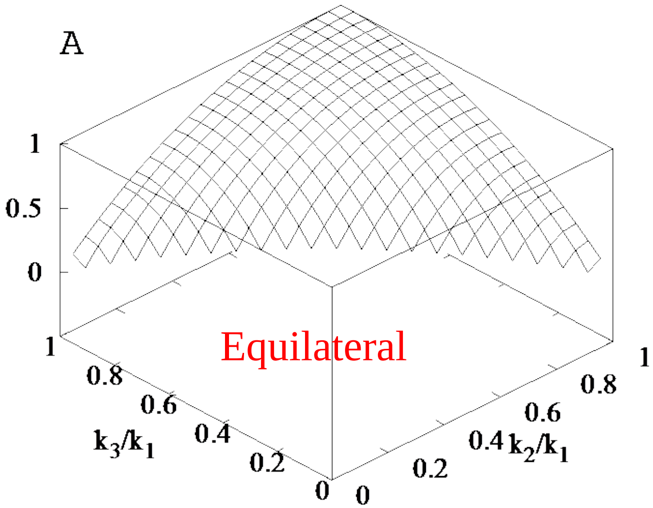
(iii) $G_5(X) = \mu X^2/M^{10}$

$$\mathcal{A}_{\mathcal{R}}^{\rm lead} \simeq (-0.012/c_s^2) S_7^{\rm equil} + (0.016/c_s^2) S_7^{\rm ortho}$$

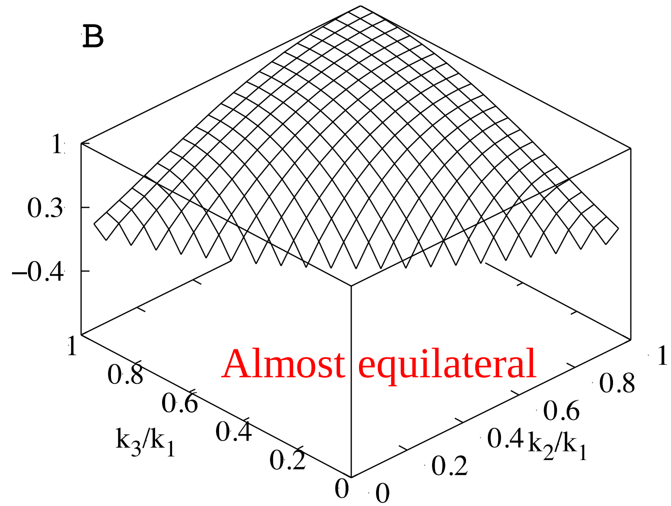
The orthogonal contribution dominates.

$$C^{\rm equil} = -0.165, \quad C^{\rm ortho} = 0.888 \quad \text{for } c_s^2 = 0.01$$

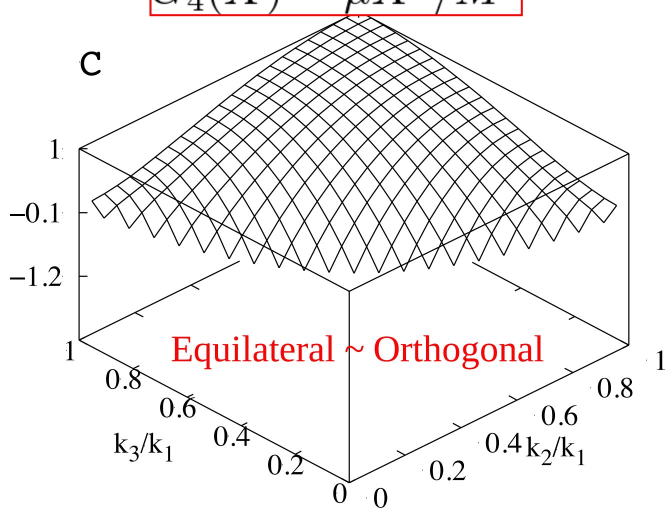
Power-law k-inflation



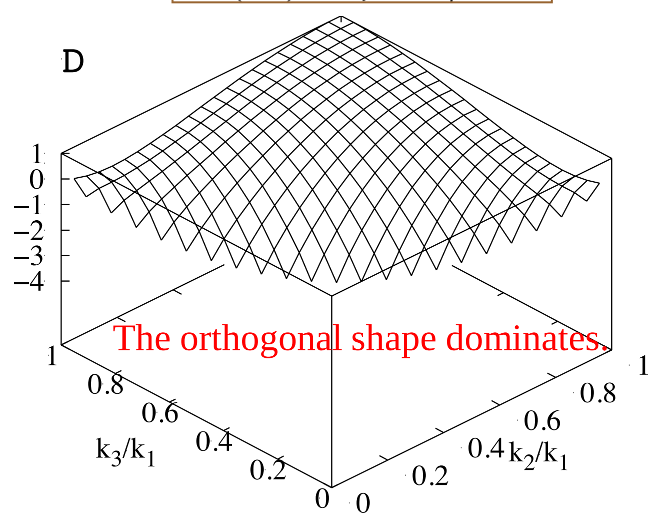
$G_3(X) = \mu X/M^4$



$G_4(X) = \mu X^2/M^7$



$G_5(X) = \mu X^2/M^{10}$



Conclusions

1. We derived the bispectrum of scalar non-Gaussianities in the Horndeski's most general scalar-tensor theories (one scalar degree of freedom).
2. Our formula with slow-variation corrections can be used for any shape of non-Gaussianities (including the squeezed case).
3. We expressed the leading-order bispectrum in terms of equilateral and orthogonal bases.
4. There are some models in which the orthogonal contribution domianates over the equilateral one.

Future outlook

1. The Planck data will be able to constrain many inflationary models.
2. Especially, the detection of local non-Gaussianities is a challenge for single-field inflationary models.