Stability of cosmological solutions in massive gravity

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Based on collaborations with

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Outline

Classify perturbations with respect to 3d rotational symmetries:

- DOF in metric $\delta g_{\mu\nu}$:

+4 scalars | +4 vectors | +2 tensors

- $\delta g_{0\mu}$ components are non-dynamical:

-2 scalars | -2 vectors | -0 tensors

- In GR, general coordinate invariance $x^\mu \rightarrow x^\mu + \xi^\mu$:

-2 scalars | -2 vectors | -0 tensors

$\Longrightarrow$ GR has only 2 tensors (gravity waves).

- In a generic massive theory, no gauge invariance:

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- Massive spin–2 in 4d $\Rightarrow$ (1 s, 2 v, 2 t). Extra scalar (BD ghost) removed with an additional constraint, stemming from the tuning of the coefficients in the EFT.

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- Introduce four scalar fields (à la Stückelberg), one for each broken gauge degree: ϕ^a $(a = 0, 1, 2, 3)$
- Requiring Poincaré symmetry in the field space. Invariant "line element":

$$ds_\phi^2 = \eta_{ab}\, d\phi^a\, d\phi^b$$

- Mass term depends only on $g_{\mu\nu}$ and the *fiducial metric*

$$f_{\mu\nu} = \eta_{ab}\, \partial_\mu \phi^a\, \partial_\nu \phi^b$$

- Requiring that the sixth degree (BD ghost) is canceled at any order, the most general action is:

$$S_m[g_{\mu\nu}, f_{\mu\nu}] = M_p^2 m_g^2 \int d^4x \sqrt{-g}\, (\mathcal{L}_2 + \alpha_3 \mathcal{L}_3 + \alpha_4 \mathcal{L}_4)$$

$$\left[\begin{array}{ll} \mathcal{L}_2 = \frac{\epsilon^{\mu\nu\rho\sigma}\epsilon_{\alpha\beta\rho\sigma}}{2} K^\alpha{}_\mu K^\beta{}_\nu & \\ \mathcal{L}_3 = \frac{\epsilon^{\mu\nu\rho\sigma}\epsilon_{\alpha\beta\gamma\sigma}}{3!} K^\alpha{}_\mu K^\beta{}_\nu K^\gamma{}_\rho & \text{and} \quad K^\mu{}_\nu \equiv \delta^\mu{}_\nu - \left(\sqrt{g^{-1}f}\right)^\mu{}_\nu \\ \mathcal{L}_4 = \frac{\epsilon^{\mu\nu\rho\sigma}\epsilon_{\alpha\beta\gamma\delta}}{4!} K^\alpha{}_\mu K^\beta{}_\nu K^\gamma{}_\rho K^\delta{}_\sigma & \end{array}\right]$$

- dRGT theory: A massive gravity theory with 5 degrees of freedom, built partly to address the late time acceleration.
  $\Rightarrow$ Can we get cosmological solutions?
- Look for simplest solutions in the simplest version of the theory.
  $\Longrightarrow$Does it work?
  (continuity with GR, stability, description of thermal history...)
  - yes $\Rightarrow$ constrain the theory/new effects?
  - no $\Rightarrow$ relax the solution and/or theory

Massive gravity zoology in 3+1

1. Drop Poincaré symmetry in the field space

$$f_{\mu\nu} = \bar{\eta}_{ab}\,\partial_\mu\phi^a\,\partial_\nu\phi^b\,,$$

   with generic $\bar{\eta}$. Hassan, Rosen, Schmidt-May '11
2. Ghost-free bigravity: introduce dynamics for the fiducial metric Hassan, Rosen '11
3. Quasi-dilaton, varying mass, ...
   d'Amico, Gabadadze, Hui, Pirtskhalava '12
   Huang, Piao, Zhou '12

In this talk, we only allow extensions of the type 1.

[*See talk by C. Lin about cosmology in the extension of type 3*]

We want

- Homogeneous and isotropic universe solution, which can accommodate the history of the universe.
- Preserved homogeneity/isotropy for linear perturbations

- FRW ansatz for the both physical and fiducial metrics

$$ds^2 = -N(t)^2\, dt^2 + a(t)^2\, \Omega_{ij}\, dx^i\, dx^j$$
$$ds_\phi^2 = -n(\phi^0)^2\, (d\phi^0)^2 + \alpha(\phi^0)^2\, \Omega_{ij}\, d\phi^i\, d\phi^j$$

$$\left[\begin{array}{c} \Omega_{ij} = \delta_{ij} + \dfrac{K\, \delta_{il}\delta_{jm} x^l\, x^m}{1 - K\, \delta_{lm} x^l x^m} \\ \langle\phi^a\rangle = \delta^a_\mu\, x^\mu \end{array}\right]$$

Is this form for $f_{\mu\nu}$ the only choice?

- Case with different $f_{\mu\nu} \longrightarrow$ FRW $g_{\mu\nu}$

  d'Amico et al'11
  Koyama et al'11; Volkov'11,'12; Gratia, Hu, Wyman'12
  Kobayashi, Siino, Yamaguchi, Yoshida'12; Motohashi, Suyama'12
- Although background dynamics homogeneous+isotropic, there *is* a broken FRW symmetry in the Stückelberg sector, which *can* be probed by perturbations.

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Cosmological solutions
Simplest case

- Fiducial metric: Minkowski

$$ds_\phi^2 = -n(t)^2\,(dt)^2 + \delta_{ij} dx^i\, dx^j$$

- Physical metric: Flat FRW

$$ds^2 = -N(t)^2\, dt^2 + a(t)^2\, \Omega_{ij}\, dx^i\, dx^j$$

- Constraint from Stückelberg fields:

$$m_g^2\, \partial_t\, (a^3 - a^2) = 0$$

$\Longrightarrow$ No flat FRW for Minkowski reference metric.

- Way out: Inhomogeneous universe, which looks FRW-like within $1/m_g$ size patches.

`d'Amico, de Rham, Dubovsky, Gabadadze, Pirtskhalava, Tolley '11`

- Instead: we will relax the condition on spatial flatness.

- But open FRW solutions exist

$$ds^2 = -N^2\,dt^2 + a^2\,\Omega_{ij}^{(K<0)}\,dx^i\,dx^j$$

$$ds_\phi^2 = -n^2\,dt^2 + \alpha^2\,\Omega_{ij}^{(K<0)}\,dx^i\,dx^j$$

$$\left[n = \dot{\alpha}/\sqrt{|K|}\right] \Longleftarrow \text{Minkowski in open chart}$$

Minkowski in open coordinates

- Minkowski metric $\quad ds_\phi^2 = -[d\tilde{\phi}^0]^2 + \delta_{ij} d\tilde{\phi}^i d\tilde{\phi}^j$
- After coordinate transformation

$$\tilde{\phi}^0 = \frac{\alpha(\phi^0)}{\sqrt{|K|}}\sqrt{1 + |K|\delta_{ij}\phi^i\phi^j}\,, \qquad \tilde{\phi}^i = \alpha(\phi^0)\phi^i\,.$$

becomes:

$$ds_\phi^2 = -\frac{[\alpha'(\phi^0)]^2}{|K|}[d\phi^0]^2 + [\alpha(\phi^0)]^2\,\Omega_{ij}(\{\phi^i\})\,d\phi^i d\phi^j$$

- No closed FRW chart of Minkowski $\Longrightarrow$ no closed solution

Cosmological solutions for Minkowski reference metric

AEG, Lin, Mukohyama '11a

Equation of motion for $\phi^0 \Longrightarrow$ 3 branches of solutions:

$$\left(\frac{\dot{a}}{N} - \sqrt{|K|}\right) J_\phi\left(\frac{\alpha}{a}\right) = 0$$

- Branch I $\Longrightarrow \dot{a} = \sqrt{|K|}N \Longrightarrow g_{\mu\nu}$ is also Minkowski (open chart)

  $\Longrightarrow$ *No cosmological expansion!*

- Branch $\mathrm{II}_\pm \Longrightarrow J_\phi(\alpha/a) = 0$

  $$\left[J_\phi(X) \equiv 3 + 3\,\alpha_3 + \alpha_4 - 2\,(1 + 2\,\alpha_3 + \alpha_4)\,X + (\alpha_3 + \alpha_4)\,X^2\right]$$

$$\alpha = a\,X_\pm, \quad \text{with } X_\pm \equiv \frac{1 + 2\alpha_3 + \alpha_4 \pm \sqrt{1 + \alpha_3 + \alpha_3^2 - \alpha_4}}{\alpha_3 + \alpha_4} = \text{constant}$$

For $K = 0$, this branch not present. Only Branch I remains.

Extension to generic reference metric

AEG, Lin, Mukohyama '11b

- Extending the field space metric, the line elements are

$$ds^2 = -N^2\, dt^2 + a^2\, \Omega_{ij} dx^i\, dx^j$$

$$ds_\phi^2 = -n^2\, dt^2 + \alpha^2\, \Omega_{ij} dx^i\, dx^j$$

Generically, we can have spatial curvature with either sign

e.g. at the cost of introducing a new scale (H_f), de Sitter reference can be brought into flat, open and closed FRW form.

Equations of motion for $\phi^0 \Rightarrow$ 3 branches of solutions

$$(a\, H - \alpha H_f)\; J_\phi \left(\frac{\alpha}{a}\right) = 0$$

- Branch I : $a\, H = \alpha\, H_f$ $\quad \left[H_f \equiv \frac{\dot{\alpha}}{\alpha\, n} \right]$
- Branch II$_\pm$: 2 cosmological solutions

$$\alpha(t) = X_\pm\, a(t)$$

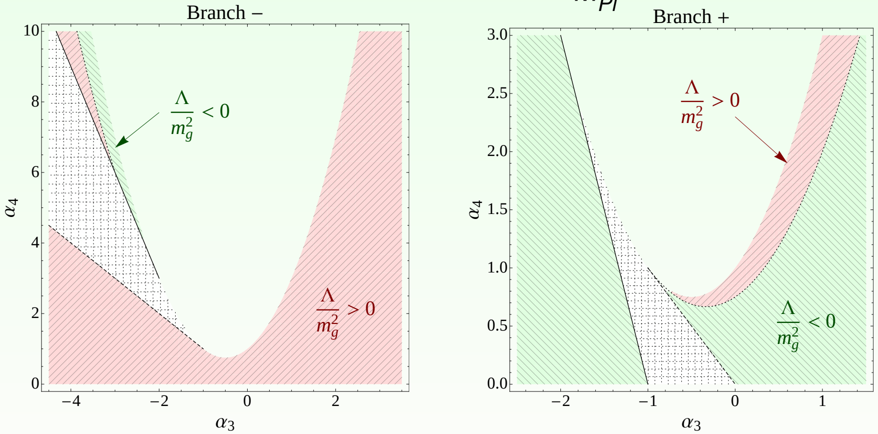
$\Longrightarrow$ same as in Minkowski reference

- Expansion in Branch I can be determined by the matter content $\Rightarrow$ in principle, can have cosmology. (Caveat: Transition from MD to Λ?)
  However, for dS reference $\Rightarrow$ Higuchi vs. Vainshtein conflict
  Fasiello, Tolley '12

Branch $\mathrm{II}_\pm$: Self-acceleration

- Evolution of Branch $\mathrm{II}_\pm$, with generic (conserved) matter

$$\left[H \equiv \frac{\dot{a}}{a\,N}\right] \longrightarrow 3\,H^2 + \frac{3\,K}{a^2} = \Lambda_\pm + \frac{1}{M_{Pl}^2}\rho \Longrightarrow \text{independent of } H_f$$



$$\Lambda_\pm \equiv -\frac{m_g^2}{(\alpha_3+\alpha_4)^2}\left[(1+\alpha_3)\left(2+\alpha_3+2\,\alpha_3^2-3\,\alpha_4\right) \pm 2\left(1+\alpha_3+\alpha_3^2-\alpha_4\right)^{3/2}\right]$$

Linear perturbations
AEG, Lin, Mukohyama '11b

- Perturbations around FRW + Generic matter fields
- Gravity sector: Since BD ghost removed, expect 5 dof.
- Decomposition wrt 3d rotations: $\boxed{1\mathrm{S} + 2\mathrm{V} + 2\mathrm{T}}$
- Quadratic action: remove auxiliary degrees, obtain:

$$S^{(2)} = \underbrace{S^{(2)}_{\mathrm{EH}} + S^{(2)}_{\mathrm{matter}} + S^{(2)}_{\Lambda_\pm}}_{\delta g_{\mu\nu}\ \text{and matter perturbations}} + \underbrace{\tilde{S}^{(2)}_{\mathrm{mass}}}_{\delta\phi^a\ \text{and}\ \delta g^{TT}_{\mu\nu}}$$

The result

- The tensor modes acquire a time dependent mass term. Dispersion relation:

$$\omega^2_{GW} = \frac{k^2}{a^2} + M^2_{GW}(t)$$

- All degrees associated with $\delta\phi^a$ (BDG+ 1S + 2V) have vanishing kinetic terms!

Cancellation of kinetic terms

Other examples of cancellation

- Self-accelerating solutions in the decoupling limit
  de Rham, Gabadadze, Heisenberg, Pirtskhalava '10
- Inhomogeneous de Sitter solutions
  Koyama, Niz, Tasinato '11
- dS and Schwarschild dS solutions in the decoupling limit
  Berezhiani, Chkareuli, de Rham, Gabadadze, Tolley '11
- A branch of self-accelerating solutions in bimetric gravity
  Crisostomi, Comelli, Pilo '12

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- dS a
- A br

What is the fate of these degrees?

- Infinitely strong coupling?
- Infinitely heavy degrees? Then, they can be integrated out $\Longrightarrow$ same d.o.f. as in GR, Higuchi bound (or its analogue) irrelevant, no need for Vainshtein mechanism.

Need to go beyond linear order to determine which case is realized

Probing the non-linear action with linear tools
de Felice, AEG, Mukohyama '12

- Symmetry of the background $\Rightarrow$ cancellation
- Instead of computing the high order action, we slightly break the isotropy and compute the quadratic terms.
- The small deviation from isotropy in the background is interpreted as a homogeneous perturbation in the FRW solution. This will allow us to obtain information on the high order terms in the exact FRW case.

Introducing small anisotropy

- The simplest anisotropic extension of flat FRW is the degenerate Bianchi type–I metric

$$ds^2 = -N^2 dt^2 + a^2 \left[e^{4\,\overset{|\sigma| \ll 1}{\sigma}} dx^2 + e^{-2\,\sigma}\,(dy^2 + dz^2)\right]$$

- Different fiducial metric $\Leftrightarrow$ different theory. In order to have continuity with the FRW solutions, we keep $f_{\mu\nu}$ isotropic:

$$ds_\phi^2 = -n^2\, dt^2 + \alpha^2\,(dx^2 + dy^2 + dz^2)$$

- Vacuum configuration (with bare Λ)

Perturbations around the anisotropic background

Strategy

- Make use of the residual symmetry on the *y*–*z* plane; decomposition wrt 2d rotations:
  (2d) $\boxed{3\mathrm{S}_{2d} + 2\mathrm{V}_{2d}}$ expected to propagate in gravity sector
- Write the quadratic action, define G.I. variables, expand fields in Fourier space, integrate out non-dynamical degrees.
- Expand background around FRW for small σ
- Diagonalize the system $\Longrightarrow$ obtain dispersion relations for energy eigenstates

Kinetic terms and eigenfrequencies

κ: Kinetic term before canonical normalization
ω: Frequency after diagonalization

2d vectors

1. $\kappa_1 = \mathcal{O}(\sigma^0) > 0$ and $\omega_1^2 = \frac{k^2}{a^2} + M_{GW}^2$
   $\Longrightarrow$ 1 of the GW in the isotropic limit
2. $\kappa_2 = \mathcal{O}(\sigma)$ and $\omega_2^2 \propto \frac{k^2}{\sigma}$
   $\Longrightarrow \kappa > 0$ if a time dependent condition satisfied

2d scalars

1. $\kappa_1 = \mathcal{O}(\sigma^0) > 0$ and $\omega_1^2 = \frac{k^2}{a^2} + M_{GW}^2$
   $\Longrightarrow$ 1 of the GW in the isotropic limit
2. $\kappa_2 = \mathcal{O}(\sigma)$ and $\omega_2^2 \propto \frac{k^2}{\sigma}$
3. $\kappa_3 = -C(\vec{k})\,\kappa_2$ and $\omega_3^2 \propto \frac{k^2}{\sigma}$, with $C(\vec{k}) > 0$

$\Longrightarrow$ Either 2 or 3 has always negative kinetic term!

There is always a ghost in 2d scalar sector. Since $\omega \propto k$, we cannot integrate it out from the low energy effective theory. The solution is unstable!

Fate of isotropic solutions?

- Quadratic kinetic term for $1 \gg |\sigma| \neq 0$
  $-(...\sigma)\,\dot{\phi}_k^2 \Longleftrightarrow \phi_{k_1}\dot{\phi}_{k_2}\dot{\phi}_{k_3}$ type terms, with $k_1 = 0$.
- Homogeneous and isotropic solutions in massive gravity have ghost instability which arises from the cubic order action.
- This conclusion is valid for $\pm$ self-accelerating branch solutions of massive gravity with arbitrary reference metric.

Avoiding the instability

Rephrasing the issue

- Rephrasing the issue: Kinetic terms of 3 degrees $\sim J_\phi\,\dot{\phi}^2$.
- Equation of motion for Stückelbergs: $(a\,H - \alpha\,H_f)J_\phi = 0$.
- On the self-accelerating branch, $J_\phi = 0$.
- Is there a way to detune the proportionality between Stückelberg EOM and kinetic terms?

Avoiding $J_\phi = 0$?

1. New solutions?

   $\Rightarrow$ Breaking the FRW symmetry of the background?
2. FRW solution in extended theory?

   $\Rightarrow$ New degree which breaks $J_\phi = 0$?
3. Partially massless theory?

   $\Rightarrow$ New symmetry to naturally remove some degrees?

Relaxing the background symmetry

Breaking the FRW symmetry in the fiducial metric

- It is still possible to have a H&I physical metric, while either H or I is broken in Stückelberg sector.
- Inhomogeneous examples already exist, but technically challenging.
- In our analysis, anisotropy introduced only as a technical tool. However, kinetic terms of these polarizations *are* second order.
  $\Longrightarrow$ A universe with finite anisotropy, which looks isotropic at the background level may have a chance to evade the ghost.

Anisotropic FRW
AEG, Lin, Mukohyama '12

- Consider Bianchi I metric, with finite anisotropy

$$ds^2 = -N^2 dt^2 + a^2 \left[e^{4\sigma} dx^2 + e^{-2\sigma} (dy^2 + dz^2) \right]$$

- Fiducial metric is de Sitter

$$ds_\phi^2 = -n^2 dt^2 + \alpha^2 \left(dx^2 + dy^2 + dz^2 \right) \longleftarrow \left[\frac{\dot{\alpha}}{\alpha\, n} = H_f = \text{constant} \right]$$

Vacuum configuration: Fixed points

- Seek solutions with $\dot{H} = \dot{X} = \dot{\sigma} = 0 \longleftarrow \left[H \equiv \frac{\dot{a}}{a\,N} \,,\; X \equiv \frac{\alpha}{a} \right]$
- Dropping isotropic F.P., and points that require fine tuning gives

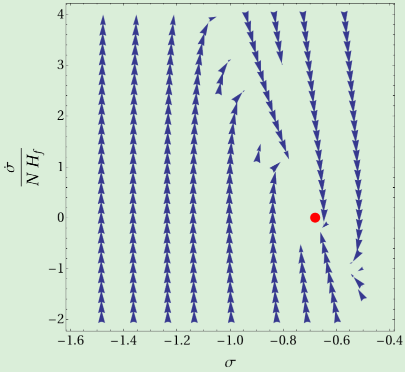
$$e^\sigma = \sqrt{\frac{H_f\, X}{H}}$$

- The remaining equations of motion reduce to algebraic equations on X and H.

Local stability

- Perturb H, σ and X around the F.P. value
- Can reduce the equations to
$$\delta\sigma'' + 3\,X_0\,e^{-2\,\sigma_0}\,\delta\sigma' + M^2\,\delta\sigma = 0 \longleftarrow \left[\,' \equiv \frac{1}{H_f\,N}\,\frac{d}{dt}\right]$$
- Local stability requirement: $M^2(\frac{m_g}{H_f},\,\alpha_3,\,\alpha_4) > 0$

Global Stability



- Parameters:
  $m_g = 20\,H_f\,,\ \alpha_3 = -\frac{1}{20}\,,\ \alpha_4 = 1$
- Fixed point:
  $X \simeq 4\,,\ e^{\sigma} \simeq \frac{1}{2}\,,\ H \simeq 16\,H_f$
- On F.P., isotropic expansion $\dot\sigma = 0$. In GR, this is equivalent to a FRW universe. In MG, a coordinate redefinition renders physical metric isotropic, but now the fiducial metric becomes anisotropic.
  $\Rightarrow$ Anisotropic FRW

Anisotropic FRW - Stability of perturbations

De Felice, AEG, Lin, Mukohyama '13

- Equation of motion for Stückelberg sector:

$$\left(H + 2\,\dot{\sigma} - H_f\, e^{-2\,\sigma}\, X\right)\, J_\phi^{(x)} + 2\,\left(H - \dot{\sigma} - H_f\, e^{\sigma}\, X\right)\, J_\phi^{(y)} = 0$$

- The factorized form is broken with anisotropy. In the isotropic limit $\sigma \to 0$, we recover: $(H - H_f\, X) J_\phi = 0$.

On the fixed point, $e^\sigma = \sqrt{H_f\, X / H}$, $\dot{\sigma} = 0$

- 2 modes (1 in 2dv, 1 in 2ds) have non-vanishing kinetic terms, which are positive for a range of parameters.
- However, E.O.M. still sets $J_\phi^{(y)} = 0$.
- Kinetic terms for 3 modes arising from mass term:

$$2 \text{ modes} \propto J_\phi^{(y)}\,, \qquad 1 \text{ mode} \propto J_\phi^{(x)}\,,$$

- Hence, on the fixed point, we still have 2 modes with vanishing kinetic terms.

- As we did for FRW background, we can introduce homogeneous deformations of the background to obtain information on non-linear terms.

$$\sigma = \sigma_0 + \sigma_1$$

- The kinetic terms of two modes are order $\mathcal{O}(\sigma_1)$, and are equal up to numerical factors.

$$\kappa = (...)\sigma_1 + (...)\dot{\sigma}_1 + \mathcal{O}(\sigma^2)$$

- Kinetic terms depend on 2 dynamical functions!
- On the other hand, depending on the initial conditions, kinetic terms may be positive throughout the evolution.

- Equation of motion for σ_1:

$$\ddot{\sigma}_1 + 3\,H_0\,\dot{\sigma}_1 + M_\sigma^2\,\sigma_1 = 0$$

- Over-damping $\Longrightarrow 9H_0^2 > 4\,M_\sigma^2$
- Late time attractor

$$\dot{\sigma}_1 \longrightarrow \left(-\frac{3}{2}\,H_0 + \sqrt{\frac{9\,H_0^2}{4} - M_\sigma^2}\right)\sigma_1$$

- So, if the system is close to the attractor,

$$\kappa = (...)\sigma_1$$

  and if σ_1 has the correct sign, the kinetic terms of the two modes can be positive throughout the evolution.
- However, the two modes with vanishing kinetic terms on the fixed point do not have a mass gap. This signals that these modes are strongly coupled.
- Nevertheless, this is the first example of a homogeneous cosmology in dRGT which is stable.

Summary

- dRGT theory admits self-accelerating, H & I universe solutions, but these suffer from a non-linear instability. This conclusion is valid for any reference metric.
- New solution with finite anisotropy – but purely isotropic expansion. Background equivalent to FRW, anisotropy in the Stückelberg sector can be probed by metric perturbations. Expect the breaking of statistical isotropy to be subdominant by the smallness of m_g.
- The ghost instability may be avoided for a range of parameters and initial conditions. However, two degrees in anisotropic background suffer from strong coupling. Other backgrounds with different form of anisotropy?
- Extending the theory with new degrees of freedom may render the perturbations stable.

Bonus Slide: Gauge invariant variables - FRW

AEG, Lin, Mukohyama '11b

- Perturbations in the metric, Stückelberg fields and matter fields:

$$g_{00} = -N^2(t)\left[1+2\phi\right]\,, \quad g_{0i} = N(t)a(t)\beta_i\,, \quad g_{ij} = a^2(t)\left[\Omega_{ij}(x^k) + h_{ij}\right]$$

$$\varphi^a = x^a + \pi^a + \frac{1}{2}\pi^b\partial_b\pi^a + O(\epsilon^3)\,, \qquad \sigma_I = \sigma_I^{(0)} + \delta\sigma_I \longleftarrow \text{matter sector}$$

- Scalar-vector-tensor decomposition:

$$\beta_i = D_i\beta + S_i\,, \qquad \pi_i = D_i\pi + \pi_i^T\,,$$

$$h_{ij} = 2\psi\Omega_{ij} + \left(D_iD_j - \tfrac{1}{3}\Omega_{ij}\triangle\right)E + \tfrac{1}{2}(D_iF_j + D_jF_i) + \gamma_{ij}$$

$$\left\}\begin{array}{c} D_i \leftarrow \Omega_{ij}\,, \quad \triangle \equiv \Omega^{ij}D_iD_j \\ D^iS_i = D^i\pi_i^T = D^iF_i = 0 \\ D^i\gamma_{ij} = \gamma^i_i = 0 \end{array}\right.$$

- Gauge invariant variables without Stückelberg fields:

Originate from $g_{\mu\nu}$ and matter fields $\delta\sigma_I$

$$\begin{aligned} Q_I &\equiv \delta\sigma_I - \mathcal{L}_Z\sigma_I^{(0)}\,, \\ \Phi &\equiv \phi - \tfrac{1}{N}\partial_t(NZ^0)\,, \\ \Psi &\equiv \psi - \tfrac{\dot a}{a}Z^0 - \tfrac{1}{6}\triangle E\,, \\ B_i &\equiv S_i - \tfrac{a}{2N}\dot F_i\,, \end{aligned} \qquad \left(\begin{array}{c} Z^0 \equiv -\frac{a}{N}\beta + \frac{a^2}{2N^2}\dot E \\ Z^i \equiv \frac{1}{2}\Omega^{ij}(D_jE + F_j) \\ \text{Under } x^\mu \to x^\mu + \xi^\mu : \\ Z^\mu \to Z^\mu + \xi^\mu \end{array}\right)$$

- However, we have 4 more degrees of freedom:

Associated with Stückelberg fields

$$\psi^\pi \equiv \psi - \frac{1}{3}\triangle\pi - \frac{\dot a}{a}\pi^0\,, \qquad E^\pi \equiv E - 2\,\pi\,, \qquad F_i^\pi \equiv F_i - 2\,\pi_i^T$$

Bonus slide: Quadratic action - FRW

- After using background constraint for Stückelberg fields:

$$S^{(2)} = \underbrace{S^{(2)}_{\rm EH} + S^{(2)}_{\rm matter} + S^{(2)}_{\Lambda_\pm}}_{\text{depend only on } Q_I,\Phi,\Psi,B_i,\gamma_{ij}} + \underbrace{\tilde{S}^{(2)}_{\rm mass}}_{\tilde{S}^{(2)}_{\rm mass}=S^{(2)}_{\rm mass}-S^{(2)}_{\Lambda_\pm}}$$

- The first part is equivalent to GR + $\Lambda_\pm$+ Matter fields σ_I.
- The additional term:

$$M^2_{GW} \equiv m_g^2 \left(1 - \frac{H}{X_\pm H_f}\right) X_\pm^2 \times [(1+2\alpha_3+\alpha_4) - X_\pm(\alpha_3+\alpha_4)]$$

$$\tilde{S}^{(2)}_{\rm mass} = M_p^2 \int d^4x\, N\, a^3 \sqrt{\Omega}\, M^2_{GW} \times \left[3(\psi^\pi)^2 - \frac{1}{12}E^\pi \triangle(\triangle+3K)E^\pi + \frac{1}{16}F^i_\pi(\triangle+2K)F_i^\pi - \frac{1}{8}\gamma^{ij}\gamma_{ij}\right]$$

- The only common variable is γ_{ij}. Dispersion relation of tensor modes:

$$\omega^2_{GW} = \frac{k^2}{a^2} + M^2_{GW}(t)$$

- E^π, ψ^π, F_i^π have no kinetic term!

Bonus slide: Decomposition of perturbations -Bianchi I

- Perturbations are decomposed with respect to the 2d rotational symmetry around the x axis

$$
\begin{array}{rcl}
 & \begin{array}{ccc} \quad 0 \quad & \quad 1 \quad & \quad j \quad \end{array} & \\
\delta g_{\mu\nu} = \begin{array}{c} 0 \\ 1 \\ i \end{array} & \begin{pmatrix} -2\,N^2\,\Phi & a\,N\,\partial_x\chi & b\,N\,(\partial_j B + v_j) \\ & a^2\,\psi & a\,b\,\partial_x\,(\partial_j\beta + \lambda_j) \\ & & b^2\left[\tau\,\delta_{ij} + 2\,E_{,ij} + h_{(i,j)}\right] \end{pmatrix} & \\
\delta\phi^\mu = & \begin{pmatrix} \pi^0 & \partial_x\pi^1 & \partial^i\pi + \pi^i \end{pmatrix} &
\end{array}
\qquad
\left[\begin{array}{c} i,j=2,3 \\ \partial^i v_i = 0 \\ \partial^i\lambda_i = 0 \\ \partial^i h_i = 0 \\ \partial^i \pi_i = 0 \end{array}\right]
$$

- Advantage of the axisymmetry: 2d scalars and 2d vectors decouple at linear level.
- Physical degrees in 2d scalar sector (*even modes*)
  (10 total) - (3 non-dynamical) - (3 gauge) - (1 BD ghost) = 3
- Physical degrees in 2d vector sector (*odd modes*)
  (4 total) - (1 non-dynamical) - (1 gauge) = 2
- We are interested in the stability of the gravity sector, so we do not include any matter fields. Only bare Λ.

Bonus slide: Gauge invariant variables -Bianchi I

GI constructed only out of $\delta g_{\mu\nu}$

$$
\begin{aligned}
\hat\Phi &= \Phi - \frac{1}{2\,N}\,\partial_t\left(\frac{\tau}{H_b}\right)\\
\hat\chi &= \chi + \frac{1}{2\,a\,H_b}\tau - \frac{a}{N}\,\partial_t\left[\frac{b}{a}\,\left(\beta - \frac{b}{2\,a}\,E\right)\right]\\
\hat B &= B + \frac{1}{2\,b\,H_b}\,\tau - \frac{b}{2\,N}\,\partial_t E\\
\hat\psi &= \psi - \frac{H_a}{H_b}\,\tau - \frac{b}{a}\,\partial_x^2\,\left(2\,\beta - \frac{b}{a}\,E\right)\\
\hat v_i &= v_i - \frac{b}{2\,N}\,\partial_t h_i\\
\hat\lambda_i &= \lambda_i - \frac{b}{2\,a}\,h_i
\end{aligned}
$$

GI referring to $\delta\phi^a$

$$
\begin{aligned}
\hat\tau_\pi &= \pi^0 - \frac{\tau}{2\,N\,H_b}\\
\hat\beta_\pi &= \pi^1 - \frac{b}{a}\,\left(\beta - \frac{b}{2\,a}\,E\right)\\
\hat E_\pi &= \pi - \frac{1}{2}\,E\\
\hat h_{\pi\,i} &= \pi_i - \frac{1}{2}\,h_i
\end{aligned}
$$

Strategy

- Use gauge invariant variables to keep track of the new massive graviton degrees. This removes the pure gauge combinations.
- Integrate out non-dynamical degrees (4 in the 2d scalar sector, 1 in the 2d vector sector)
- Expand around FRW solution for small anisotropy
- Diagonalize the Lagrangian: Bring the action to the canonical form by rescaling and rotating the fields. $\Longrightarrow$ $\boxed{\text{Obtain dispersion relations}}$

Bonus slide: Odd sector – 2d vectors

- The action, after small anisotropy expansion, takes the form:

$$S^{(2)}_{\rm odd} \simeq \frac{M_{\rm Pl}^2}{2}\int N\,dt\,dk_L d^2k_T \bar{a}^3\left[K_{11}\frac{|\dot{\mathcal{Q}}_1|^2}{N^2} - \Omega_{11}^2\,|\mathcal{Q}_1|^2 + K_{22}\frac{|\dot{\mathcal{Q}}_2|^2}{N^2} - \Omega_{22}^2\,|\mathcal{Q}_2|^2\right]$$

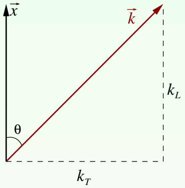
at leading order:

$$K_{11} = \frac{k_L^2\,k_T^4}{2\,k^2} \qquad K_{22} = \frac{\bar{a}^2\,k_T^2\,M_{GW}^2}{4\left(1-\frac{\bar{a}^2\,n^2}{\alpha^2\,N^2}\right)}\,\sigma$$

$$\frac{\Omega_{11}^2}{K_{11}} = \frac{k^2}{\bar{a}^2} + M_{GW}^2 \qquad \frac{\Omega_{22}^2}{K_{22}} = \frac{1}{2\,\sigma}\left(1-\frac{\bar{a}^2\,n^2}{\alpha^2\,N^2}\right)\frac{k^2}{\bar{a}^2}$$

1 GW in FRW

New degree



- condition for avoiding the ghost and gradient instability:

$$\left(1-\frac{\bar{a}\,n}{\alpha\,N}\right)\sigma > 0$$

Bonus slide: Even sector – 2d scalars

- The full quadratic action is formally (in terms of G.I. quantities)

$$S^{(2)}_{\text{even}} = \frac{M_p^2}{2} \int N\, dt\, dk_L\, d^2k_T\, a\, b^2 \mathcal{L}_{\text{even}}$$

$$\mathcal{L}_{\text{even}} = \frac{\dot{\mathcal{Y}}^\dagger}{N} K \frac{\dot{\mathcal{Y}}}{N} - \mathcal{Y}^\dagger \Omega^2 \mathcal{Y} + \mathcal{Z}^\dagger \mathcal{A} \mathcal{Y} + \mathcal{Y}^\dagger \mathcal{A}^T \mathcal{Z} + \mathcal{Z}^\dagger \mathcal{B} \frac{\dot{\mathcal{Y}}}{N} + \frac{\dot{\mathcal{Y}}^\dagger}{N} \mathcal{B}^T \mathcal{Z} + \mathcal{Z}^\dagger \mathcal{C} \mathcal{Z}$$

- $\mathcal{Y} \Rightarrow$ 3 dynamical degrees (in GR, 2 are gauge)
- $\mathcal{Z} \Rightarrow$ 4 non-dynamical degrees (including the BD ghost $\pi^0 - \frac{\tau}{2\,N\,H_b}$)
- E.O.M. for n.d. modes

$$\mathcal{Z} = -\mathcal{C}^{-1} \left(\mathcal{A} \mathcal{Y} + \mathcal{B} \frac{\dot{\mathcal{Y}}}{N} \right)$$

- Now all 3 d.o.f in the action are dynamical

$$\mathcal{L}_{\text{even}} = \frac{\dot{\mathcal{Y}}^\dagger}{N} \bar{K} \frac{\dot{\mathcal{Y}}}{N} + \frac{\dot{\mathcal{Y}}^\dagger}{N} \bar{M} \mathcal{Y} + \mathcal{Y}^\dagger \bar{M}^T \frac{\dot{\mathcal{Y}}}{N} - \mathcal{Y}^\dagger \bar{\Omega}^2 \mathcal{Y}$$

$$\left[\bar{K} = K - \mathcal{B}^T \mathcal{C}^{-1} \mathcal{B}\,, \qquad \bar{M} = -\mathcal{B}^T \mathcal{C}^{-1} \mathcal{A}\,, \qquad \bar{\Omega}^2 = \Omega^2 + \mathcal{A}^T \mathcal{C}^{-1} \mathcal{A} \right]$$

- Use small anisotropy expansion and diagonalize $\bar{K}$ at leading order

1 GW in FRW $\leftarrow$ $\kappa_1 = \frac{k_T^4}{8\,k^4}$

$$\kappa_2 = -\frac{2\,\bar{a}^2\, M_{GW}^2\, k_L^2}{\left(1 - \frac{\bar{a}^2\, n^2}{\alpha^2\, N^2}\right)} \sigma$$

$$\kappa_3 = - \frac{k_T^2}{2\,k_L^2} \kappa_2$$

wrong sign!