On toric Weierstrass models: Work in progress

Antonella Grassi, Vittorio Perduca

University of Pennsylvania

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Credits: We used the computer program *Sage* to first test our conjectures.

1. Toric Geometry
   - Calabi-Yau, Elliptic Fibration
   - Calabi Yau as hypersurfaces in toric varieties
2. Elliptically fibered Calabi Yau
   - $K3$ toric elliptic
3. Toric Weierstrass models
   - Review: Weierstrass models
   - Toric Weierstrass model
   - Semistable polytopes
   - Sections
   - Applications

Toric geometry

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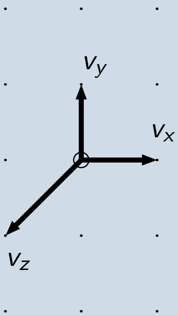
From fans, via homogeneous coordinates

To every fan Σ in $N \simeq \mathbb{Z}^n$, a lattice, one associates X_Σ of dimension n

Example

$\mathbb{P}^2$



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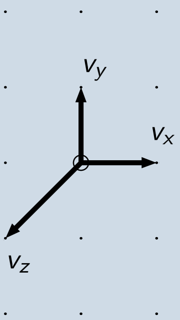

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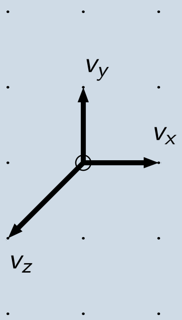

- $(v_x, v_y, v_z) \leftrightarrow (x, y, z)$ "homogeneous coordinates"
- Define:

$$X_\Sigma := \mathbb{C}^3 - Z_\Sigma / \sim$$

with $Z_\Sigma = \{\mathbf{0}\}$ and quotient action:

$$(x, y, z) \sim (\lambda^{q_x} x, \lambda^{q_y} y, \lambda^{q_z} z) = (\lambda x, \lambda y, \lambda z),$$

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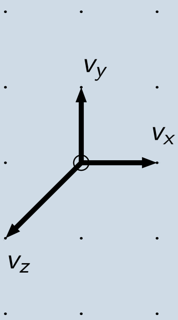

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- Z_Σ, G, quotient action (q_i), are determined by the fan.

Calabi-Yau, Elliptic Fibration

V is a <u>Calabi-Yau variety</u> if $K_V \sim \mathcal{O}(V)$, $h^i(\mathcal{O}(V)) = 0,\ 0 < i < dimV$.

$dimV = 1,\ V$: is an elliptic curve, T^2, cubic in $\mathbb{P}^2$.

$dimV = 2,\ V$: is a $K3$ surface, e, g, quartic in $\mathbb{P}^3$

$\pi_V : V \to B_V$ is an <u>elliptic fibration with section</u> $\leftrightarrow \pi_V^{-1}(p)$ is a elliptic curve with a marked point.

Toric divisors

Fact

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$-K_{X_\Sigma} = \sum D_i$, D_i invariant toric divisors.

Calabi Yau as hypersurfaces in Fano toric varieties

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- $\Delta \ \leftrightarrow \ (\mathbb{P}_\Delta, \mathcal{L}_\Delta)$, where $\mathbb{P}_\Delta := X_\Sigma$ is a toric variety over the fan Σ on the faces of ∇, and

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- If $z_1, \ldots, z_N$ are the homogeneous coordinates, $\mathcal{L}_\Delta$ is generated by

$$\{z_1^{\langle v_1,\omega\rangle+1} z_2^{\langle v_2,\omega\rangle+1} \cdot \ldots \cdot z_N^{\langle v_N,\omega\rangle+1}\}, \ \forall \omega \in \Delta \cap M$$

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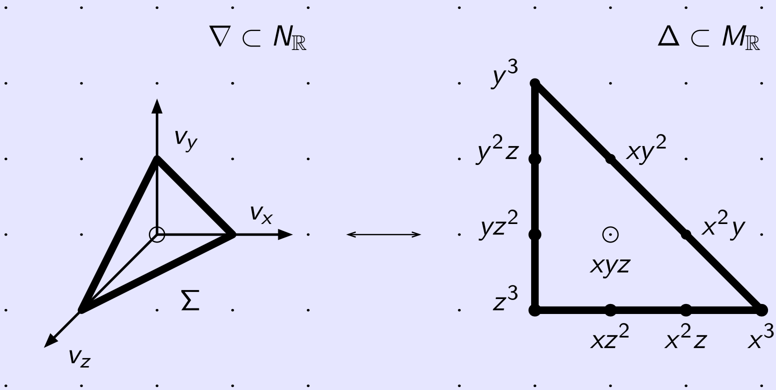

$\mathbb{P}^2_{[x,y,z]}$ $\qquad$ $(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(3))$

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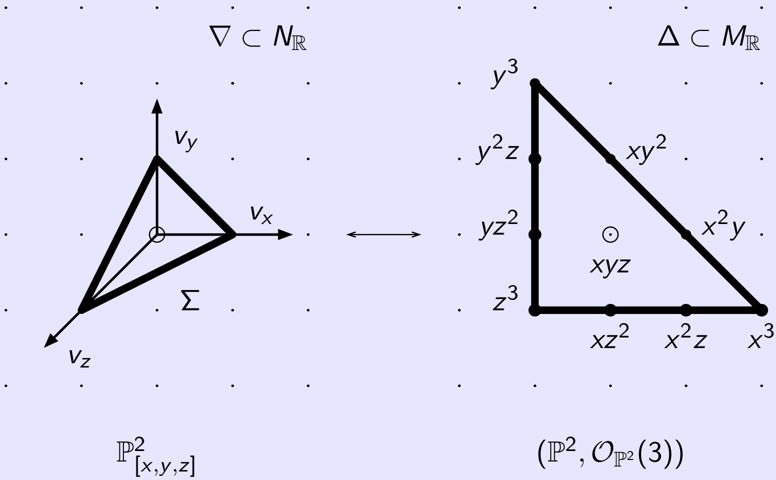

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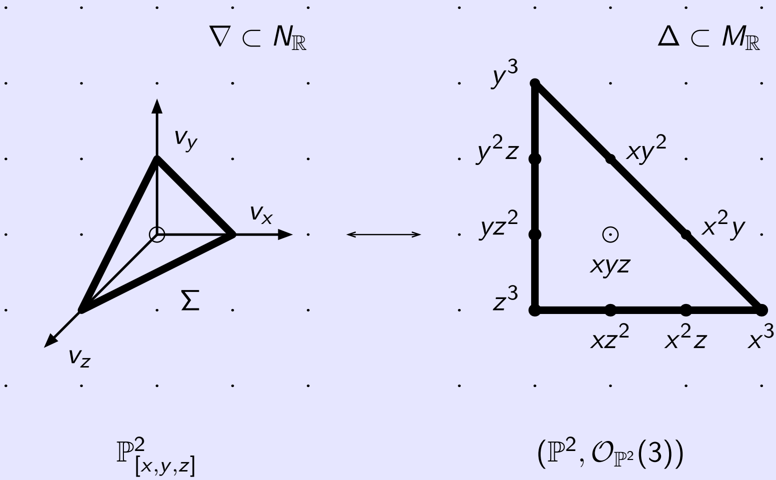

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Note: $D_x + D_y + D_z = -K_{\mathbb{P}^2}$, V elliptic curve, Calabi Yau

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$\pi : \mathbb{P}^1 \times \mathbb{P}^1_{[z_1,z_2,z_3,z_4]} \to \mathbb{P}^1_{[z_3,z_4]}$,
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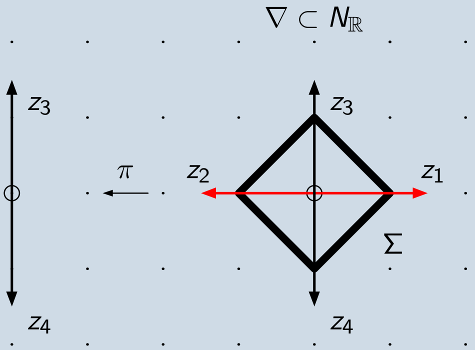

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- $X \to \mathbb{P}^1$ is fibered by two points.
  Two points in $\mathbb{P}^1$ are a 0-dim Calabi-Yau.

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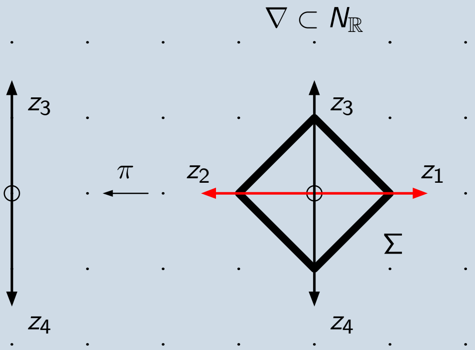

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From now on: $K3$ toric elliptic

Assume:

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- $\pi : V \to \mathbb{P}^1$, is an elliptic fibred $K3$.

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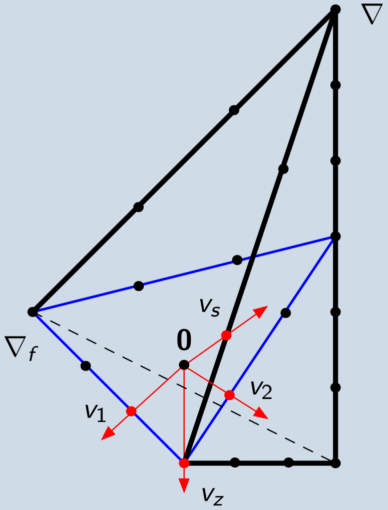

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We describe the combinatorial properties of ∇, Δ which characterize the Weierstrass toric models.

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Know (Nakayama): Given $\pi : X \to B$, *elliptic*, with section $\sigma : B \to X$, the Weierstrass model $\pi_W : W \to B$ is birational to X:

$$\begin{tikzcd} X \arrow[rr] \arrow[dr, "\pi"'] & & W \arrow[dl, "\pi_W"] \\ & B & \end{tikzcd}$$

Nakayama: $W \hookrightarrow \mathbf{P} = \mathbb{P}(\mathcal{O}_B \oplus \mathcal{L}^2 \oplus \mathcal{L}^3)$, where $\mathcal{L} \simeq p_*\mathcal{O}_T(T)$.

If X is a $K3$ surface, $B \simeq \mathbb{P}^1$, and $\mathcal{L} \simeq \mathcal{O}_{\mathbb{P}^1}(-2)$:

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W is described by the equation $y^2 = x^3 + ax + b$, $a(s,t)$ $b(s,t)$ homogeneous, $\deg = 8, 12$.

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Let us see some examples:

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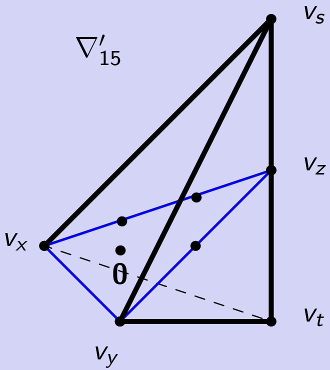

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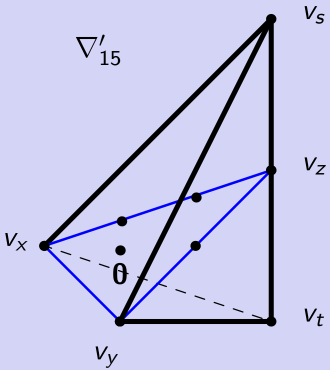

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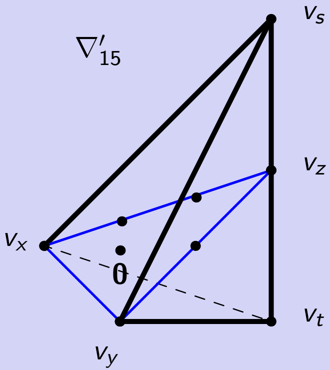

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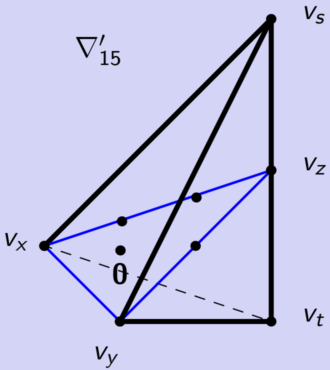

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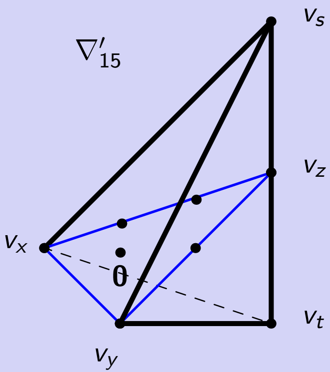

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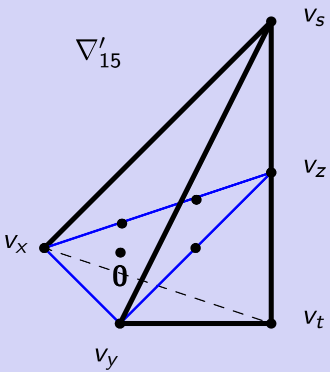

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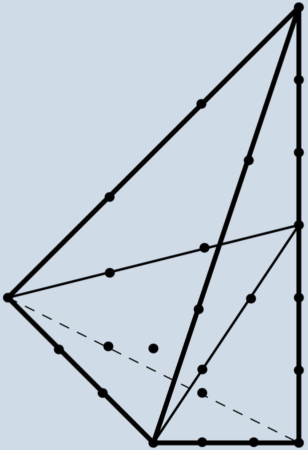

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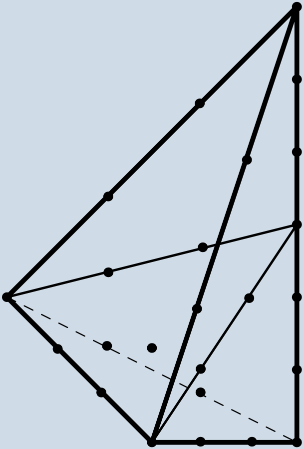

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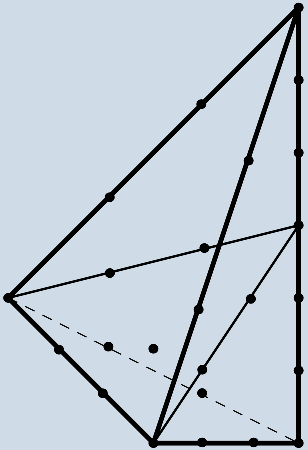

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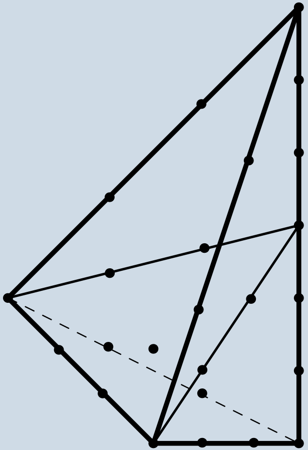

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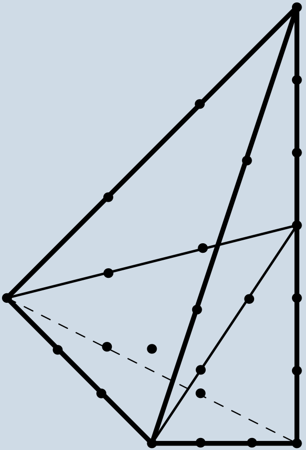

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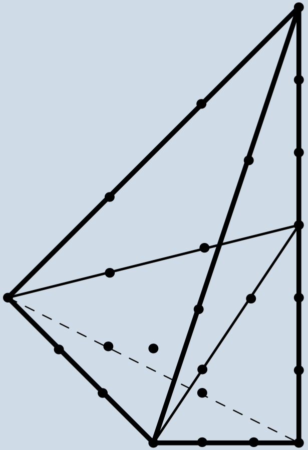

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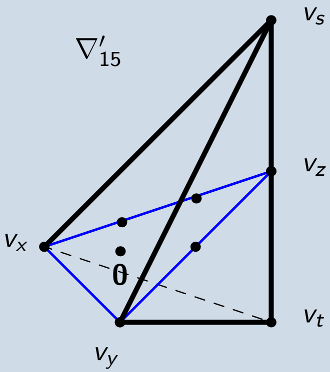

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Can we identify a section from the combinatorics?

Fix v_z, assume it is a section:

Definition

$W \subset X_\Sigma$ is a Weierstrass toric model $\longleftrightarrow$

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[The "stick" in Candelas-Font]

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Fact 4. Any $V \subset X_\Sigma$ has a toric Weierstrass model $\longleftrightarrow$ Δ_{X_Σ} is a subpolytope of Δ_W, some W semistable.

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Example: (Next page)

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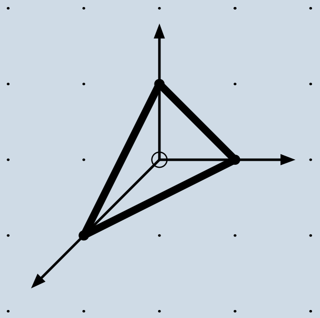
Take two ∇f, reflexive, as below:



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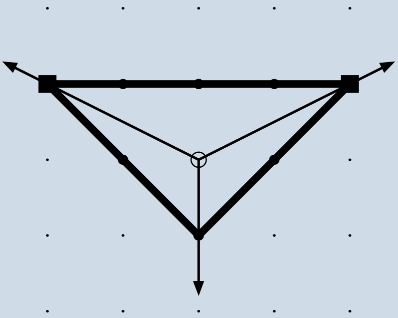
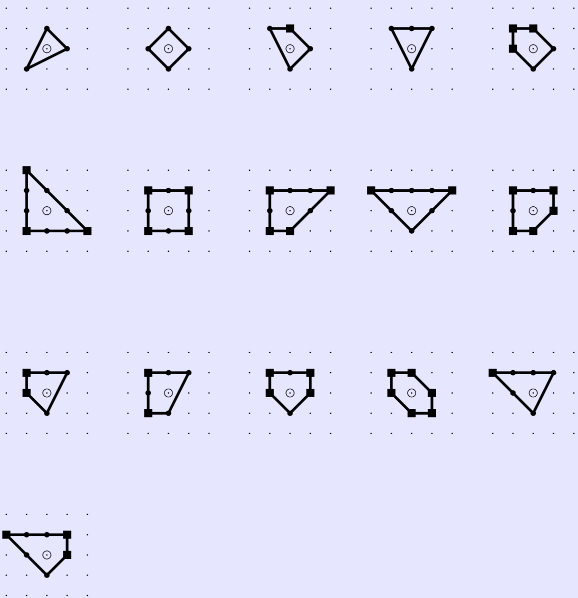


$\nabla_f : \mathbb{P}^{(2,1,1)}/\mathbb{Z}_2$

■ = irreducible toric section

All the irreducible sections for the 16 (up to $SL(2,\mathbb{Z})$) two-dimensional reflexive polytopes :



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Arithmetic & Physics:

- Compute: Mordell Weil lattice of sections
- Find: Torsion sections
- Use: degenerations of $K3$ to rational elliptic surfaces (which arise also in $F-$theory-Heterotic Duality)

Applications...

In progress:

- *Toric* Jacobian of elliptic toric fibration without a sections
- Higher dimension: Calabi-Yau threefold, fourfolds.
- Compute height of sections.
- Is a Toricall Weierstrass model unique (torically)?
- Find the "Narrow" MW lattice.