Higher Derivative and Conformal Gravity from Bimetric and Partially Massless Bimetric Theories

Fawad Hassan

Stockholm University, Sweden

IPMU Workshop on "Massive gravity and its cosmological implications"

Tokyo, Japan, April 8-10, 2013

In Collaboration with

- Rachel A. Rosen,
  arXiv:1103.6055, 1106.3344, 1109.3515, 1109.3230,1111.2070

- Angnis Schmidt-May & Mikael von Strauss
  arXiv:1203.5283, 1204.5202,1208:1515, 1208:1797, 1212:4525, 1303.6940

Outline of the talk

Review: Linear and Nonlinear massive spin-2 fields

Ghost-free bi-metric theory

Mass spectrum of bimetric theory

Partially Massless bimetric theory

Higher drivative gravity from bimetric theory

Equivalence between CG and PM bimetric theory

Outline of the talk

Review: Linear and Nonlinear massive spin-2 fields

Ghost-free bi-metric theory

Mass spectrum of bimetric theory

Partially Massless bimetric theory

Higher drivative gravity from bimetric theory

Equivalence between CG and PM bimetric theory

Linear massive spin-2 fields

The Fierz-Pauli equation:

Linear massive spin-2 field, $h_{\mu\nu}$, in background metric $\bar{g}_{\mu\nu}$

$$\bar{\mathcal{E}}^{\rho\sigma}_{\mu\nu}\, h_{\rho\sigma} - \Lambda\Big(h_{\mu\nu} - \tfrac{1}{2}\bar{g}_{\mu\nu}h^{\rho}_{\rho}\Big) + \frac{m_{\rm FP}^2}{2}\,\left(h_{\mu\nu} - \bar{g}_{\mu\nu}h^{\rho}_{\rho}\right) = 0$$

[Fierz-Pauli, 1939]

- 5 propagating modes (massive spin-2)
- Massive gravity?
- What determines $\bar{g}_{\mu\nu}$? (flat, dS, AdS, $\cdots$)
- Nonlinear generalizations?

[The Boulware-Deser ghost (1972)]

Nonlinear generalizations of FP theory

- Massive gravity (fixed $f_{\mu\nu}$):

$$\mathcal{L} = m_p^2 \sqrt{-g} \left[R - m^2 \, V(g^{-1} f) \right]$$

- Massive spin-2 field + gravity (dynamical f):

$$\mathcal{L} = m_p^2 \sqrt{-g} \left[R - m^2 \, V(g^{-1} f) \right] + \mathcal{L}(\nabla f)(?)$$

Bimetric: $\mathcal{L}(\nabla f) = m_f^2 \sqrt{-f} \, R_f (?)$

[Isham-Salam-Strathdee, 1971, 1977]

Generically, both contain a $\mathcal{GHOST}$ at the nonlinear level

[Boulware-Deser, 1972]

Counting modes:

Generic massive gravity:

- Linear: 5 modes
- Non-linear: 6 modes (massive spin-2 + ghost)

Generic bimetric theory:

- Linear: 5 modes (massive, $(\delta g - \delta f)$)
  2 mode (massless, $(\delta g + \delta f)$)
- Non-linear: 7 modes + 1 (ghost)

Complication: Since the ghost shows up nonlinearly, its absence needs to be established nonlinearly

Construction of ghost-free nonlinear theories

Based on "Decoupling limit" (perturbative):

A specific $V(\sqrt{g^{-1}\eta})$ was obtained and shown to be ghost-free in a "decoupling limit" and also perturbatively in $h = g - \eta$

[de Rham, Gabadadze, 2010; de Rham, Gabadadze, Tolley, 2010]

Non-linear Hamiltonian methods (non-perturbative):

Construction of ghost-free nonlinear theories

Based on "Decoupling limit" (perturbative):

A specific $V(\sqrt{g^{-1}\eta})$ was obtained and shown to be ghost-free in a "decoupling limit" and also perturbatively in $h = g - \eta$

[de Rham, Gabadadze, 2010; de Rham, Gabadadze, Tolley, 2010]

Non-linear Hamiltonian methods (non-perturbative):

Questions not answerable by "decoupling limit":

- Is massive gravity with $V(\sqrt{g^{-1}\eta})$ ghost-free nonlinearly?

  [SFH, Rosen (1106.3344, 1111.2070)]

- Is it ghost-free for generic fixed $f_{\mu\nu}$?

  [SFH, Rosen, Schmidt-May (1109.3230)]

  (see Cedric's talk for alternative approaches)

- Can $f_{\mu\nu}$ be given ghost-free dynamics?

  [SFH, Rosen (1109.3515)]

Outline of the talk

Review: Linear and Nonlinear massive spin-2 fields

Ghost-free bi-metric theory

Mass spectrum of bimetric theory

Partially Massless bimetric theory

Higher drivative gravity from bimetric theory

Equivalence between CG and PM bimetric theory

Ghost-free bimetric theory

Digression: Elementary symmetric polynomials of $\mathbb{X}$ with eigenvalues $\lambda_1, \cdots, \lambda_4$:

$$
\begin{aligned}
e_0(\mathbb{X}) &= 1, \qquad\qquad e_1(\mathbb{X}) = \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4\,, \\
e_2(\mathbb{X}) &= \lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_1\lambda_4 + \lambda_2\lambda_3 + \lambda_2\lambda_4 + \lambda_3\lambda_4, \\
e_3(\mathbb{X}) &= \lambda_1\lambda_2\lambda_3 + \lambda_1\lambda_2\lambda_4 + \lambda_1\lambda_3\lambda_4 + \lambda_2\lambda_3\lambda_4\,, \\
e_4(\mathbb{X}) &= \lambda_1\lambda_2\lambda_3\lambda_4 = \det\mathbb{X}\,.
\end{aligned}
$$

$$
\begin{aligned}
e_0(\mathbb{X}) &= 1\,, \qquad\qquad e_1(\mathbb{X}) = [\mathbb{X}]\,, \\
e_2(\mathbb{X}) &= \tfrac{1}{2}([\mathbb{X}]^2 - [\mathbb{X}^2]), \\
e_3(\mathbb{X}) &= \tfrac{1}{6}([\mathbb{X}]^3 - 3[\mathbb{X}][\mathbb{X}^2] + 2[\mathbb{X}^3])\,, \\
e_4(\mathbb{X}) &= \tfrac{1}{24}([\mathbb{X}]^4 - 6[\mathbb{X}]^2[\mathbb{X}^2] + 3[\mathbb{X}^2]^2 + 8[\mathbb{X}][\mathbb{X}^3] - 6[\mathbb{X}^4])\,, \\
e_k(\mathbb{X}) &= 0 \qquad \text{for} \quad k > 4\,,
\end{aligned}
$$

$$[\mathbb{X}] = \mathrm{Tr}(\mathbb{X})\,, \qquad e_n(\mathbb{X}) \sim (\mathbb{X})^n$$

- The $e_n(\mathbb{X})$'s and $\det(\mathbb{1}+\mathbb{X})$:

$$\det(\mathbb{1}+\mathbb{X}) = \sum\nolimits_{n=0}^{4} e_n(\mathbb{X})$$

$$= \sum_{n=0}^{4} \frac{-1}{n!(4-n)!}\, \epsilon_{\mu_1\cdots\mu_n\lambda_{n+1}\cdots\lambda_4}\, \epsilon^{\nu_1\cdots\nu_n\lambda_{n+1}\cdots\lambda_4}\, \mathbb{X}^{\mu_1}{}_{\nu_1} \cdots \mathbb{X}^{\mu_n}{}_{\nu_n}$$

- Introduce "deformed determinant" :

$$\widehat{\det}(\mathbb{1}+\mathbb{X}) = \sum\nolimits_{n=0}^{4} \beta_n\, e_n(\mathbb{X})$$

- Observation:

$$V(\sqrt{g^{-1}f}) = \widehat{\det}(\mathbb{1}+\sqrt{g^{-1}f}\,)$$

[SFH & R. A. Rosen (1103.6055)]

Ghost-free bi-metric theory

Ghost-free combination of kinetic and potential terms for g & f:

$$\mathcal{L} = m_g^2 \sqrt{-g} R_g - 2m^4 \sqrt{-g} \sum_{n=0}^{4} \beta_n \, e_n(\sqrt{g^{-1} f}) + m_f^2 \sqrt{-f} \, R_f$$

Note,

$$\sqrt{-g} \sum_{n=0}^{4} \beta_n \, e_n(\sqrt{g^{-1} f}) = \sqrt{-f} \sum_{n=0}^{4} \beta_{4-n} \, e_n(\sqrt{f^{-1} g})$$

Hamiltonian analysis: 7 nolinear propagating modes, no ghost!

$$C(\gamma, \pi) = 0 \,, \quad C_2(\gamma, \pi) = \tfrac{d}{dt} C(x) = \{H, C\} = 0$$

[SFH, Rosen (1109.3515,1111.2070)]

Outline of the talk

Review: Linear and Nonlinear massive spin-2 fields

Ghost-free bi-metric theory

Mass spectrum of bimetric theory

Partially Massless bimetric theory

Higher drivative gravity from bimetric theory

Equivalence between CG and PM bimetric theory

Mass spectrum of bimetric theory

[SFH, A. Schmidt-May, M. von Strauss 1208:1515, 1212:4525]

$$S_{gf} = -\int d^d x \left[m_g^{d-2} \sqrt{g} R_g - 2 m^d \sqrt{g} \sum_{n=0}^{d} \beta_n \, e_n(S) + m_f^{d-2} \sqrt{f} R_f \right]$$

Equations of motion:

$$R_{\mu\nu}(g) - \tfrac{1}{2} g_{\mu\nu} R(g) + V^g_{\mu\nu} = T^g_{\mu\nu}$$
$$R_{\mu\nu}(f) - \tfrac{1}{2} f_{\mu\nu} R(f) + V^f_{\mu\nu} = T^f_{\mu\nu}$$

(for classical solutions, see talk by Mikhail Volkov)

- When are the 7 fluctuations in $\delta g_{\mu\nu}$, $\delta f_{\mu\nu}$ good mass eigenstates? (well-defined FP mass)
- How to characterize deviations from General Relativity?

Mass spectrum: proportional backgrounds

FP masses exist only around,

$$\bar{f}_{\mu\nu} = c^2 \bar{g}_{\mu\nu}$$

g and f equations:

$$R_{\mu\nu}(\bar{g}) - \tfrac{1}{2}\bar{g}_{\mu\nu}R(\bar{g}) + \begin{pmatrix}\Lambda_g \\ \Lambda_f\end{pmatrix}\bar{g}_{\mu\nu} = 0 \text{ or } \begin{pmatrix}T^g_{\mu\nu} \\ T^f_{\mu\nu}\end{pmatrix}$$

$$\Lambda_g = \frac{m^d}{m_g^{d-2}} \sum_{k=0}^{d-1} \binom{d-1}{k} c^k \beta_k, \quad \Lambda_f = \frac{m^d}{m_f^{d-2}} \sum_{k=1}^{d} \binom{d-1}{k-1} c^{k+2-d} \beta_k$$

Implication:

$$\Lambda_g = \Lambda_f \quad \Rightarrow \quad c = c(\beta_n, \alpha \equiv m_f/m_g)$$

(Exception: Partially massless (PM) theory)

Mass spectrum around proportional backgrounds

Linear modes:

$$\delta M_{\mu\nu} = \tfrac{1}{2c}\left(\delta f_{\mu\nu} - c^2 \delta g_{\mu\nu}\right)\,, \qquad \delta G_{\mu\nu} = \left(\delta g_{\mu\nu} + \alpha^{d-2} c^{d-4} \delta f_{\mu\nu}\right)$$

$$\bar{\mathcal{E}}_{\mu\nu}^{\rho\sigma}\,\delta G_{\rho\sigma} - \Lambda_g\big(\delta G_{\mu\nu} - \tfrac{1}{2}\bar{g}_{\mu\nu}\bar{g}^{\rho\sigma}\delta G_{\rho\sigma}\big) = 0\,,$$

$$\bar{\mathcal{E}}_{\mu\nu}^{\rho\sigma}\,\delta M_{\rho\sigma} - \Lambda_g\big(\delta M_{\mu\nu} - \tfrac{1}{2}\bar{g}_{\mu\nu}\bar{g}^{\rho\sigma}\delta M_{\rho\sigma}\big) + \tfrac{1}{2} m_{\rm FP}^2 \left(\delta M_{\mu\nu} - \bar{g}_{\mu\nu}\bar{g}^{\rho\sigma}\delta M_{\rho\sigma}\right) = 0$$

The FP mass of δM:

$$m_{\rm FP}^2 = \frac{m^d}{m_g^{d-2}}\left(1 + (\alpha c)^{2-d}\right)\sum_{k=1}^{d-1}\binom{d-2}{k-1} c^k \beta_k$$

Bimetric as massive spin-2 field + gravity

Nonlinear extensions of linear modes:

$$G_{\mu\nu} = g_{\mu\nu} + c^{d-4}\alpha^{d-2} f_{\mu\nu}\,, \quad M^G_{\mu\nu} = G_{\mu\rho}\left(\sqrt{g^{-1}f}\right)^{\rho}{}_{\nu} - cG_{\mu\nu}$$

G: No ghost-free matter coupling, not the gravitational metric.

- Bimetric as gravity + massive spin-2 field:

$$g_{\mu\nu}, \qquad M_{\mu\nu} = g_{\mu\rho}\left(\sqrt{g^{-1}f}\right)^{\rho}{}_{\nu} - cg_{\mu\nu}\,, \qquad m_g >> m_f$$

(see talk by Keisuke Izumi)

- $M = 0 \Rightarrow$ GR.
  $M \neq 0 \Rightarrow$ deviations from GR, driven by matter couplings.

Outline of the talk

Review: Linear and Nonlinear massive spin-2 fields

Ghost-free bi-metric theory

Mass spectrum of bimetric theory

Partially Massless bimetric theory

Higher drivative gravity from bimetric theory

Equivalence between CG and PM bimetric theory

Partial masslessness in FP theory

$$\bar{\mathcal{E}}_{\mu\nu}^{\rho\sigma}\, h_{\rho\sigma} - \Lambda\big(h_{\mu\nu} - \tfrac{1}{2}\bar{g}_{\mu\nu}h^{\rho}_{\rho}\big) + \frac{m^2_{\rm FP}}{2}\,\big(h_{\mu\nu} - \bar{g}_{\mu\nu}h^{\rho}_{\rho}\big) = 0$$

dS/Einstein backgrounds:

$$\bar{g}_{\mu\nu}: \qquad R_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu} = 0$$

Higuchi Bound:

$$m^2_{FP} = \tfrac{2}{3}\Lambda$$

New gauge symmetry:

$$\Delta h_{\mu\nu} = (\nabla_\mu\nabla_\nu + \tfrac{\Lambda}{3})\xi(x)$$

Gives 5-1=4 propagating modes

[Deser, Waldron, $\cdots$ (1983-2012)]

Can a nonlinear extension of PM theory exist?

Partial masslessness beyond FP theory

Non-linear PM theory = Nonlinear spin-2 fields with a gauge invariance!

Does it exist? Independent of dS/Einstein backgrounds?

Partial masslessness beyond FP theory

Non-linear PM theory = Nonlinear spin-2 fields with a gauge invariance!

Does it exist? Independent of dS/Einstein backgrounds?

Known perturbative results around dS:

- Cubic PM vertices ($\sim h^3$) in $d = 4$ *[Zinoviev (2006)]*
- Cubic PM vertices exist only in $d = 3, 4$ with 2 derivatives
  For $d > 4$, higher derivative theory needed.
  [Joung, Lopez, Taronna (2012)]

We will identify a specific bimetric theory as the candidate nonlinear PM theory

Partial masslessness in Bimetric theory

[SFH, Schmidt-May, von Strauss, 1208:1797, 1212:4525]

1) Assume a nonlinear bimetric theory with PM symmetry exists

2) Around $\bar{f} = c^2 \bar{g}$, $\delta M_{\mu\nu} \sim h_{\mu\nu}$ satisfies the FP equation. Then the action of symmetry on $\delta M_{\mu\nu}$ & $\delta G_{\mu\nu}$ must be:

$$\delta M_{\mu\nu} \to \delta M_{\mu\nu} + \left(\nabla_\mu \nabla_\nu + \frac{\Lambda}{3}\, \bar{g}_{\mu\nu}\right)\xi(x)\,, \qquad \delta G_{\mu\nu} \to \delta G_{\mu\nu}$$

Partial masslessness in Bimetric theory

[SFH, Schmidt-May, von Strauss, 1208:1797, 1212:4525]

1) Assume a nonlinear bimetric theory with PM symmetry exists

2) Around $\bar{f} = c^2 \bar{g}$, $\delta M_{\mu\nu} \sim h_{\mu\nu}$ satisfies the FP equation. Then the action of symmetry on $\delta M_{\mu\nu}$ & $\delta G_{\mu\nu}$ must be:

$$\delta M_{\mu\nu} \to \delta M_{\mu\nu} + \Big(\nabla_\mu \nabla_\nu + \tfrac{\Lambda}{3}\, \bar{g}_{\mu\nu}\Big)\xi(x)\,, \qquad \delta G_{\mu\nu} \to \delta G_{\mu\nu}$$

- Find the transformation of $\delta g_{\mu\nu}$ & $\delta f_{\mu\nu}$.
- Shift the transf. to dynamical backgrounds $\bar{g}_{\mu\nu}$ & $\bar{f}_{\mu\nu}$
- For the dS-preserving subset $\xi = \xi_0$ (cont), this gives,

$$\bar{g}'_{\mu\nu} = (1 + a\xi_0\,)\bar{g}_{\mu\nu}\,, \quad \bar{f}'_{\mu\nu} = (1 + b\xi_0\,)\bar{f}_{\mu\nu}$$

$$\bar{f}' = c'^2(\xi_0)\,\bar{g}' \qquad c' \neq c$$

A symmetry can exist only if $\Lambda_g = \Lambda_f$ does not determine c

Candidate PM bimetric theory in d=4

The necessary condition for the existence of PM symmetry is that c is not determined by $\Lambda_g = \Lambda_f$, or

$$\beta_1 + \left(3\beta_2 - \alpha^2\beta_0\right) c + \left(3\beta_3 - 3\alpha^2\beta_1\right) c^2 + \left(\beta_4 - 3\alpha^2\beta_2\right) c^3 + \alpha^2\beta_3 c^4 = 0$$

This gives the candidate nonlinear PM theory (d=4)

$$\alpha^2\beta_0 = 3\beta_2\,, \qquad 3\alpha^2\beta_2 = \beta_4\,, \qquad \beta_1 = \beta_3 = 0$$

Nonlinear PM bimetric theory

Checks:

- $$m_{\rm FP}^2 = 2\frac{m^4}{m_g^2}\left(\alpha^{-2}+c^2\right)\beta_2 = \frac{2}{3}\Lambda_g$$
- For $d>4$, all $\beta_n=0$. Nonlinear PM bimetric exists only for $d=3,4$.
- In $d>4$ PM is restored by Lanczos-Lovelock terms
- Realization of the ξ_0 gauge transformation in the nonlinear theory.

Full Gauge symmetry of the nonlinear theory? (not yet known, but ....)

Outline of the talk

Review: Linear and Nonlinear massive spin-2 fields

Ghost-free bi-metric theory

Mass spectrum of bimetric theory

Partially Massless bimetric theory

Higher drivative gravity from bimetric theory

Equivalence between CG and PM bimetric theory

Higher derivative gravity and Conformal gravity

HD gravity:

$$S^{\mathrm{HD}}_{(2)}[g] = m_g^2 \int d^4x \sqrt{g} \left[\Lambda + c_R R(g) - \frac{c_{RR}}{m^2} \left(R^{\mu\nu} R_{\mu\nu} - \frac{1}{3} R^2 \right) \right]$$

7 modes: 2 (massless spin-2) + 5 (massive spin-2 ghost)
[Stelle (1977)]

Coformal Gravity:

$$S^{\mathrm{CG}}[g] = -c \int d^4x \sqrt{g} \left[R^{\mu\nu} R_{\mu\nu} - \frac{1}{3} R^2 \right]$$

Invariant under Weyl scalings $\Rightarrow$
6 modes: 2 (massless spin-2) + 4 ghost modes
[Riegert (1984), Maldacena (2011)]

Conformal gravity and PM theory

- Maldacena: CG spectrum in dS background $\sim$ linear PM. CG nonlinear PM candidate?
- Deser-Waldron: No PM spectrum away from dS backgrounds (caveate: too restrictive condition on the spectrum)

HD gravity from Bimetric theory

Define

$$S = \sqrt{g^{-1} f}\,, \qquad P_{\mu\nu} = R_{\mu\nu} - \frac{1}{2(d-1)} g_{\mu\nu} R$$

HD gravity from Bimetric theory

Define

$$S = \sqrt{g^{-1}f}\,, \qquad P_{\mu\nu} = R_{\mu\nu} - \frac{1}{2(d-1)} g_{\mu\nu} R$$

Solve the bimetric $g_{\mu\nu}$ equation algebraically for $f_{\mu\nu}$, as an expansion in $R_{\mu\nu}(g)/m^2$,

$$S^\mu_{\ \nu} = a\delta^\mu_\nu + \frac{a_1}{m^2} P^\mu_{\ \nu} + \frac{a_2}{m^4}\left[\left({P^\mu_{\ \nu}}^2 - PP^\mu_{\ \nu}\right) + \frac{1}{d-1} e_2(P)\delta^\mu_\nu\right] + \mathcal{O}(m^{-6})$$

Compute $f = f(g)$. Then

$$S^{\mathrm{BM}}[g, f(g)] = S^{\mathrm{HD}}[g]$$

HD gravity from Bimetric theory

- 4-derivative ($\sim R^2$) truncation:

$$S^{\mathrm{BM}}_{(2)}[g, f(g)] = S^{\mathrm{HD}}_{(2)}[g]$$

  The spin-2 ghost in 4-derivative HD gravity is an artifact of this truncation (can be illustrated in a linear theory).

- The correspondence is not an equivalence of the truncated theories (in general). Different truncated EoM's.
- For PM bimetric theory one obtains conformal gravity.

Outline of the talk

Review: Linear and Nonlinear massive spin-2 fields

Ghost-free bi-metric theory

Mass spectrum of bimetric theory

Partially Massless bimetric theory

Higher drivative gravity from bimetric theory

Equivalence between CG and PM bimetric theory

Equivalence between CG and PM bimetric theory

- CG equation of motion: The Bach equation,

$$B_{\mu\nu} = 0$$

Propagates 6 modes due to conformal invariance.

Equivalence between CG and PM bimetric theory

- CG equation of motion: The Bach equation,

$$B_{\mu\nu} = 0$$

  Propagates 6 modes due to conformal invariance.
- In PM bimetric theory, determine $f_{\mu\nu}$ from g-equation. Substitute in f-equation (not in the action) to get,

$$B_{\mu\nu} + \mathcal{O}(R^3/m^2) = 0$$

  In the low curvature limit, PM bimetric theory has a gauge symmetry even away from dS and definitely propagates $7 - 1 = 6$ modes! None is a ghost

Equivalence between CG and PM bimetric theory

- CG equation of motion: The Bach equation,

$$B_{\mu\nu} = 0$$

  Propagates 6 modes due to conformal invariance.

- In PM bimetric theory, determine $f_{\mu\nu}$ from g-equation. Substitute in f-equation (not in the action) to get,

$$B_{\mu\nu} + \mathcal{O}(R^3/m^2) = 0$$

  In the low curvature limit, PM bimetric theory has a gauge symmetry even away from dS and definitely propagates $7 - 1 = 6$ modes! None is a ghost

- CG eom is the low curvature limit of PM bimetric eom. Conversely, PM bimetric is a ghost-free completion of CG