Is there acausality in massive gravity?

-An Analysis of Characteristics In Non-Linear Massive Gravity-

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Motivation

In **Phys. Rev. Lett. 110, 111101 (Deser and Waldron)**

Superluminal propagation

Acausality

Based on Characteristics analysis

not every

On ~~any~~ hypersurface

Characteristics equation is satisfied

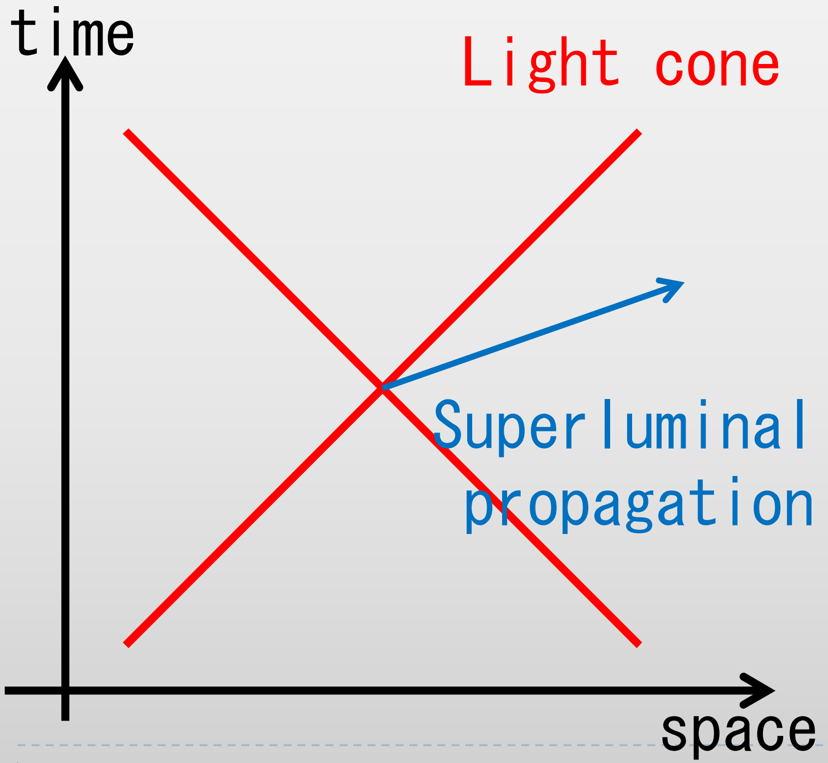
and thus information can propagate to any direction even if it is spacelike.

We revisited this problem and obtain different result

Superluminality and Acausality

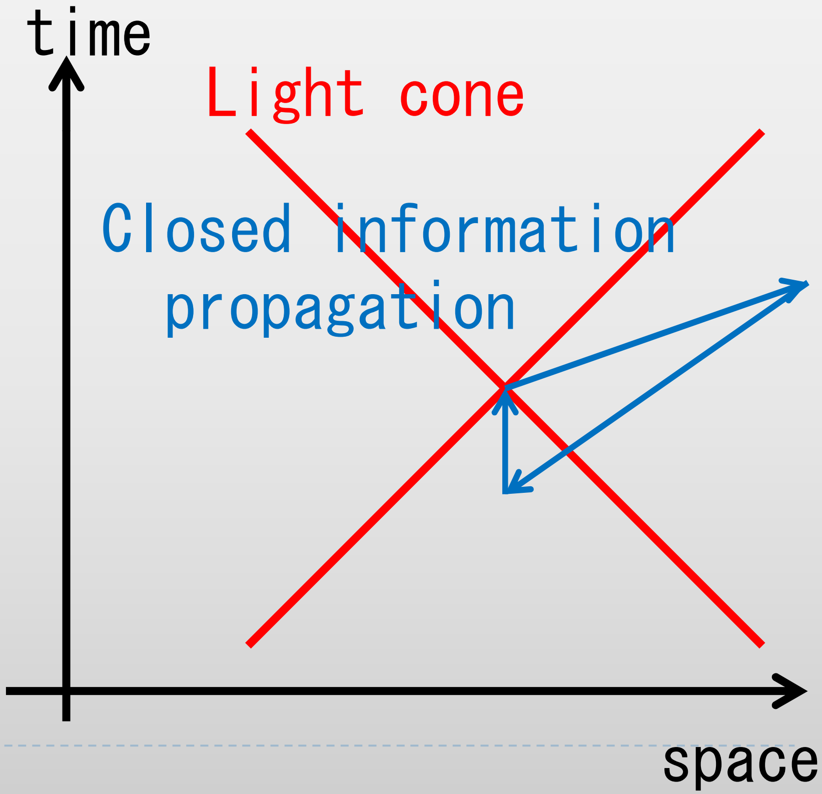
Superluminality

Propagation the speed of which is higher than that of light



Acausality

Pathological causal structure



Characteristics (Single field ODE)

Quasi-linear n-th order derivative equation

$$f[\phi, \partial_t\phi, \cdots, (\partial_t)^{n-1}\phi](\partial_t)^n\phi + g[\phi, \partial_t\phi, \cdots, (\partial_t)^{n-1}\phi] = 0$$

Time evolution

1, Initial condition; $\phi, \partial_t\phi, \cdots, (\partial_t)^{n-1}\phi$ at $t = t_0$

If $f[\phi, \partial_t\phi, \cdots, (\partial_t)^{n-1}\phi] \neq 0$

2, EoM $\Rightarrow$ $(\partial_t)^n\phi|_{t=t_0}$

($(\partial_t)^n\phi|_{t=t_0}$ is uniquely fixed)

3, $(\partial_t)^{k-1}\phi[t = t_0 + \Delta t] = (\partial_t)^k\phi[t = t_0]\Delta t + (\partial_t)^{k-1}\phi[t = t_0]$

$(1 \leq k \leq n)$

Characteristics (Single field ODE)

Quasi-linear ...h order derivative equation

Characteristics

$$f[\phi, \partial_t\phi, \cdots [\phi, \partial_t\phi, \cdots, (\partial_t)^{n-1}\phi] = 0$$

Time evolution

1, Initial condition; $\phi, \partial_t\phi, \cdots, (\cdots t = t_0$

$\boxed{\text{If } f[\phi, \partial_t\phi, \cdots, (\partial_t)^{n-1}\phi] = 0}$

2, EoM $\cancel{\Rightarrow}$ $(\partial_t)^n\phi|_{t=t_0}$

($(\partial_t)^n\phi|_{t=t_0}$ can be arbitrary)

3, $(\partial_t)^{k-1}\phi[t = t_0 + \Delta t] = (\partial_t)^k\phi[t = t_0]\Delta t + (\partial_t)^{k-1}\phi[t = t_0]$

$(1 \leq k \leq n-1)$

$(\partial_t)^{n-1}\phi[t = t_0 + \Delta t] =??$

Characteristics (Multi field ODE)

Quasi-linear 1st order derivative equations for n scalar fields

$$\sum_{j=1}^{n} f_{ij}[\phi_1, \cdots, \phi_n]\partial_t \phi_j + g_i[\phi_1, \cdots, \phi_n] = 0 \quad (1 \le i \le n)$$

- $\det f_{ij} \neq 0$
  - ➡ We can obtain the solution for all $\partial_t \phi_i$ s
  - ➡ Solution is unique
- $\det f_{ij} = 0 \Longrightarrow$ Characteristics

  A linear combination of $\partial_t \phi_i$ can be arbitrary

Characteristics (Single field PDE)

Quasi-linear n-th order derivative equation

$$f^{\nu_1\cdots\nu_n}[\phi,\partial_\mu\phi,\cdots,\partial_{\mu_1}\cdots\partial_{\mu_{n-1}}\phi]\partial_{\nu_1}\cdots\partial_{\nu_n}\phi+g[\phi,\partial_\mu\phi,\cdots,\partial_{\mu_1}\cdots\partial_{\mu_{n-1}}\phi]=0$$

Time evolution

1, Initial condition on initial hypersurface Σ

$$\phi,\partial_t\phi,\cdots,(\partial_t)^{n-1}\phi \qquad \Sigma : t=t_0$$

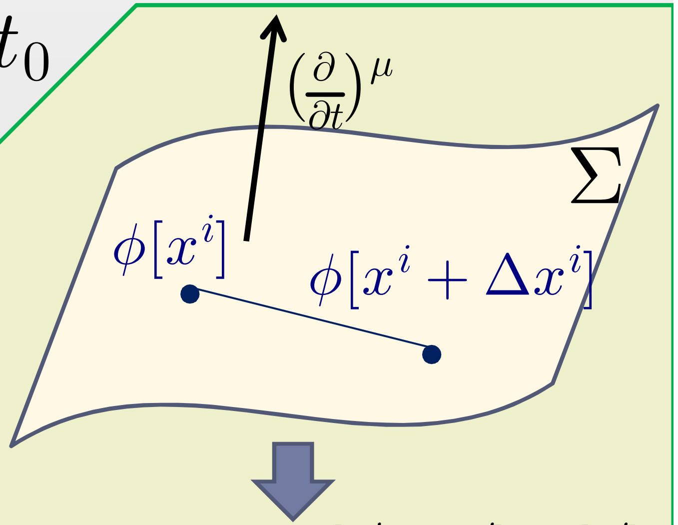
2, $\partial_i\phi,\partial_t\partial_i\phi,\partial_i\partial_j\phi,\cdots$

If $f^{t\cdots t}[\phi,\partial_\mu\phi,\cdots,\partial_{\mu_1}\cdots\partial_{\mu_{n-1}}\phi]\neq 0$

3, EoM $\Rightarrow$ $(\partial_t)^n\phi|_{t=t_0}$

($(\partial_t)^n\phi|_{t=t_0}$ is uniquely fixed)

Time evolution is unique



$$\partial_i\phi=\lim_{\Delta x^i\to 0}\frac{\phi[x^i+\Delta x^i]-\phi[x^i]}{\Delta x^i}$$

Characteristics (Single field PDE)

Quasi-linear n-th order derivative equation

$$f^{\nu_1\cdots\nu_n}[\phi,\partial_\mu\phi,\cdots,\partial_{\mu_1}\cdots\partial_{\mu_{n-1}}\phi]\partial_{\nu_1}\cdots\partial_{\nu_n}\phi+g[\phi,\partial_\mu\phi,\cdots,\partial_{\mu_1}\cdots\partial_{\mu_{n-1}}\phi]=0$$

Time ev

1, Initial … hypersurface Σ

$$\phi,\partial_t\phi,\cdots,(\partial_t)$$

2, $\partial_i\phi,\partial_t\partial_i\phi,\partial_i\partial_j\phi,\cdots$

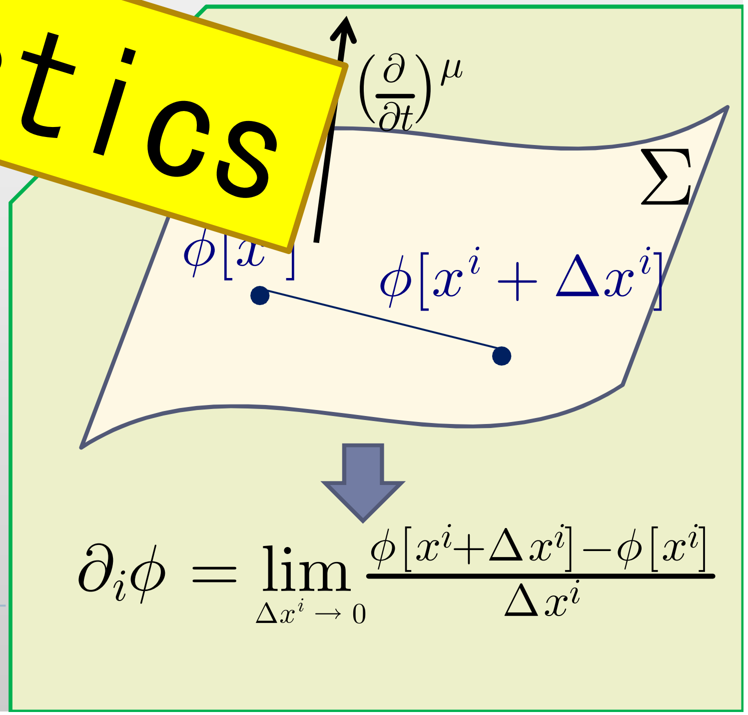
If $\boxed{f^{t\cdots t}[\phi,\partial_\mu\phi,\cdots,\partial_{\mu_1}\cdots\partial_{\mu_{n-1}}\phi]=0}$

3, EoM $\cancel{\Rightarrow}$ $(\partial_t)^n\phi|_{t=t_0}$

($(\partial_t)^n\phi|_{t=t_0}$ can be arbitrary)

Time evolution is not unique

Characteristics



2nd order PDE $\rightarrow$ 1st order PDE

Cauchy Kovalevskaya theorem :

theorem of unique time evolution

Quasi-linear 2nd order derivative equation

Evolution equation for ϕ

$u := \partial_t \phi$

$v_i := \partial_i \phi$

Trivially nonzero

$\partial_t v_i = \partial_i u$

Evolution equation for v_i

$$f^{tt}[\phi, u, v_i]\partial_t u + 2 f^{ti}\partial_i u + f^{ij}\partial_i v_j + g[\phi, u, v_i] = 0$$

Evolution equation for u

Nontrivial coefficient

$$\mathbf{U} := \{\phi, u, v_i\}$$

$$\mathbf{M}[\mathbf{U}] \cdot \partial_t \mathbf{U} = \mathbf{N}[\mathbf{U}] \cdot \partial_i \mathbf{U} + \mathbf{V}[\mathbf{U}]$$

Characteristic equation: $\det \mathbf{M}[\mathbf{U}] = 0$

Characteristics (single → multi field)

In single field

Characteristic equation is

(*Highest order derivatives w.r.t "t"*) = 0

$$f^{\nu_1 \cdots \nu_n}[\phi, \partial_\mu \phi, \cdots, \partial_{\mu_1} \cdots \partial_{\mu_{n-1}} \phi] \partial_{\nu_1} \cdots \partial_{\nu_n} \phi + g[\phi, \partial_\mu \phi, \cdots, \partial_{\mu_1} \cdots \partial_{\mu_{n-1}} \phi] = 0$$

$$f^{t \cdots t}[\phi, \partial_\mu \phi, \cdots, \partial_{\mu_1} \cdots \partial_{\mu_{n-1}} \phi] = 0$$

In multi field

Is checking highest order derivatives enough??

Sometimes yes, but generically No

Characteristics (Multi field, simple case)

Two Quasi-linear 2nd order derivative equations

$$f^{\mu\nu}[\phi, \partial_\alpha\phi, \psi, \partial_\alpha\psi]\partial_\mu\partial_\nu\phi + g^{\mu\nu}[\phi, \partial_\alpha\phi, \psi, \partial_\alpha\psi]\partial_\mu\partial_\nu\psi + h[\phi, \partial_\alpha\phi, \psi, \partial_\alpha\psi] = 0$$

$$F^{\mu\nu}[\phi, \partial_\alpha\phi, \psi, \partial_\alpha\psi]\partial_\mu\partial_\nu\phi + G^{\mu\nu}[\phi, \partial_\alpha\phi, \psi, \partial_\alpha\psi]\partial_\mu\partial_\nu\psi + H[\phi, \partial_\alpha\phi, \psi, \partial_\alpha\psi] = 0$$

Coefficients of highest order time derivatives

$$\mathbf{M} = \begin{pmatrix} f^{tt} & g^{tt} \\ F^{tt} & G^{tt} \end{pmatrix}$$

Characteristic equation: $\det \mathbf{M}[\mathbf{U}] = 0$

Characteristics (Multi field, general case)

Two equations

$$f^{\mu\nu}[\phi, \partial_\mu\phi, \psi]\partial_\mu\partial_\nu\phi + g^\mu[\phi, \partial_\mu\phi, \psi]\partial_\mu\psi + h[\phi, \partial_\mu\phi, \psi] = 0$$

$$F[\phi, \partial_\mu\phi, \psi] = 0$$

∂_μ

$$\frac{\partial F}{\partial(\partial_\nu\phi)}\partial_\mu\partial_\nu\phi + \frac{\partial F}{\partial\psi}\partial_\mu\psi + H[\phi, \partial_\mu\phi, \psi] = 0$$

Coefficients of highest order time derivatives

$$\mathbf{M} = \begin{pmatrix} f^{tt} & 0 \\ \frac{\partial F}{\partial(\partial_t\phi)} & 0 \end{pmatrix}$$

Characteristic equation: $\det \mathbf{M}[\mathbf{U}] = 0$
is always satisfied ???

Characteristics (Multi field, general case)

Two equations

$$f^{\mu\nu}[\phi, \partial_\mu \phi, \psi]\partial_\mu\partial_\nu\phi + g^\mu[\phi, \partial_\mu\phi, \psi]\partial_\mu\psi + h[\phi, \partial_\mu\phi, \psi] = 0$$

$$F[\phi, \partial_\mu\phi, \psi] := \mathcal{F}[\phi, \partial_\alpha\phi] - \psi = 0$$

$$(f^{\mu\nu} + g^\mu \frac{\partial\mathcal{F}}{\partial(\partial_\nu\phi)})\partial_\mu\partial_\nu\phi + H[\phi, \partial_\alpha\phi] = 0$$

Characteristic equation: $(f^{tt} + g^t \frac{\partial\bar{F}}{\partial(\partial_t\phi)}) = 0$

Different from that on previous slide

Characteristics (Multi field, general case)

Two equations

$$f^{\mu\nu}[\phi,\partial_\mu\phi,\psi]\partial_\mu\partial_\nu\phi + g^\mu[\phi,\partial_\mu\phi,\psi]\partial_\mu\psi + h[\phi,\partial_\mu\phi,\psi] = 0$$

$$F[\phi,\partial_\mu\phi,\psi] = 0$$

∂_μ

$$\frac{\partial F}{\partial(\partial_\nu\phi)}\partial_\mu\partial_\nu\phi + \frac{\partial F}{\partial\psi}\partial_\mu\psi + H[\phi,\partial_\mu\phi,\psi] = 0$$

Evolution equation for ϕ

Trivially nonzero

$u := \partial_t\phi$

$v_i := \partial_i\phi$

$\partial_t v_i = \partial_i u$

Evolution equation for v_i

$f^{tt}\partial_t u + g^t\partial_t\psi + 2f^{ti}\partial$ …

$\frac{\partial F}{\partial(\partial_t\phi)}\partial_t u + \frac{\partial F}{\partial\psi}\partial_t\psi +$ …

Evolution …

Nontrivial coefficients

$$\mathbf{M} = \begin{pmatrix} f^{tt} & g^t \\ \frac{\partial F}{\partial(\partial_t\phi)} & \frac{\partial F}{\partial\psi} \end{pmatrix}$$

Our model of Massive gravity

Action

$$S = (2\kappa)^{-1} \int d^4x e (R + 2 \sum_{i=0}^{4} \alpha_i \mathcal{L}_i)$$

$$\mathcal{L}_0 = 1$$
$$\mathcal{L}_1 = f$$
$$\mathcal{L}_2 = f^2 - f_{\mu\nu} f^{\mu\nu}$$
$$\vdots$$

$$f_{\mu\nu} = e_{a\mu} f^a{}_\nu$$

$e^a{}_\mu$: dynamical tetrad

$f^a{}_\nu$: fiducial tetrad

$$f = f_{\mu\nu} g^{\mu\nu}$$

For simplicity, consider the case where $\alpha_2 = \alpha_3 = \alpha_4 = 0$

EoM

$$G_{\mu\nu} - \alpha_0 g_{\mu\nu} + \alpha_1 (f_{\nu\mu} + f g_{\mu\nu}) = 0$$

$$\Rightarrow \quad f_{\mu\nu} = f_{\nu\mu}$$

Characteristic in Massive gravity

Introducing new variable

$$M_{\mu\nu\lambda} := \partial_{[\mu} f_{\nu]\lambda}$$

$$\partial M = \partial\partial f$$

$M_{0i0} = -$ … f_{0i}

$M_{0ij} = -$ … f_{ij}

$\partial_0 M_{ij0} =$ … M_{ij0}

$\partial_0 M_{ijk} =$ … M_{ijk}

$$\mathbf{M} = \begin{array}{c} \partial_t f_{00}, \partial_t M_{0i0}, \partial_t M_{0ij} \\ \begin{pmatrix} A & B & C \\ D & E & F \\ G & H & I \end{pmatrix} \end{array}$$

EoM

Characteristic equation: $\det \mathbf{M}[\mathbf{U}] = 0$

- First order derivative equations of $f_{\mu\nu}, M_{\mu\nu\lambda}$
- Give the time evolutions of f_{00}, M_{0i0}, M_{0ij}

Summary

We can not naively extend the characteristic analysis in single field to multi field case.

$\partial_t f_{00}$ joins in the Characteristic equations, and then we have different result from Deser and Waldron's

➡ Not every hypersuface can be characteristics

In order to understand the causal structure we need the detailed analysis of Characteristic equations