Non-perturbative predictions of stochastic inflation and their dependence on initial conditions

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Beyond small perturbations during slow-roll inflation

Locally – around our world-line – slow-roll inflation has both the beginning and the end.
Globally it has no beginning and no end in the most of interesting cases – in the sense that inflating patches always exist somewhere in space and time (but outside our past and future light cones).

For sufficiently large $N = \ln\left(\frac{a_f}{a}\right)$, $< h^2 >$ becomes larger than 1.

Stochastic approach to inflation ("stochastic inflation"):

$$\hat{R}_\mu^\nu - \frac{1}{2}\delta_\mu^\nu \hat{R} = 8\pi G\, \hat{T}_\mu^\nu(\hat{g}_{\alpha\beta})$$

- not as a function of $< \hat{g}_{\alpha\beta} >$!

Leads to QFT in a stochastic background.

Stochastic inflation:
1) can deal with an arbitrary large (though sufficiently smooth) global inhomogeneity;
2) takes backreaction of created fluctuations into account;
3) goes beyond any finite order of loop corrections.
Fully developed in Starobinsky (1984,1986) though the first simplified application (but beyond the one-loop approximation) was already in Starobinsky (1982).

The first main idea: splitting of the inflaton field ϕ into a large-scale and a small-scale parts with respect to H. More exactly, the border is assumed to lie at $k = \epsilon a H$ with $\exp\left(-\frac{H^2}{|\dot{H}|}\right) \ll \epsilon \ll 1$.
Applicability conditions – the standard slow-roll ones:

$$V'^2 \ll 48\pi G V^2, \quad |V''| \ll 8\pi G V/3\,.$$

Langevin equation for the large-scale field

The second main idea: a non-commutative part of the large-scale field is very small (it is composed from decaying modes), so we may neglect it. Then the remaining part is equivalent (not equal!) to a stochastic c-number (classical) field with some distribution function.

$$\frac{d\phi}{d\tau^{(n)}} = -\frac{1}{3H^{n+1}}\frac{dV}{d\tau^{(n)}} + f\ ,$$

$$< f(\tau_1^{(n)})\, f(\tau_2^{(n)}) >= \frac{H^{3-n}}{4\pi^2}\, \delta(\tau_1^{(n)} - \tau_2^{(n)})\, .$$

The Gaussian white noise f describes the flow of small-scale linear field modes through the border $k = \epsilon a H$ to the large-scale region in the course of the universe expansion.

The time-like variables: $\tau^{(n)} = \int H^n(t, \mathbf{r})\, dt$, where $H^2 = 8\pi G V(\phi)/3$.

This is not a time reparametrization $t \to f(t)$ in GR.

Different $\tau^{(n)}$ describe different stochastic processes and even have different dimensionality.

Different "clocks" are needed to measure them:

1) $n = 0$: phase of a wave function of a massive particle $(m \gg H)$;

2) $n = 1$: scalar metric perturbations (δN formalism);

3) $n = 3$: dispersion of a light scalar field generated during inflation

$$< \chi^2 >= \frac{1}{4\pi^2} < \int H^3\, dt >= \frac{< \tau^{(3)} >}{4\pi^2}\,.$$

See F. Finelli *et al.*, Phys. Rev. D **79**, 044007 (2009) for more details.

Einstein-Smoluhovsky (Fokker-Planck) equation

$$\frac{\partial \rho}{\partial \tau} = \frac{\partial}{\partial \phi}\left(\frac{V'}{3H^{n+1}}\,\rho\right) + \frac{1}{8\pi^2}\,\frac{\partial^2}{\partial \phi^2}\left(H^{3-n}\rho\right)\,.$$

Probability conservation: $\int \rho\, d\phi = 1$.

Remarks.

- More generally, the last term can be written the form

$$\frac{1}{8\pi^2}\,\frac{\partial}{\partial \phi}\left(H^{(3-n)\alpha}\,\frac{\partial}{\partial \phi}\left(H^{(3-n)(1-\alpha)}\rho\right)\right)$$

  with $0 \le \alpha \le 1$.
  $\alpha = 0$ – Ito calculus.
  $\alpha = 1/2$ – Stratonovich calculus.
  However, keeping terms explicitly depending on α exceeds the accuracy of the stochastic approach. Thus, α may put 0.

- All results are independent of the form of a cutoff in the momentum space as far as it occurs for $k \ll aH$ $(\epsilon \ll 1)$.
- Backreaction is taken into account: $\delta T_{\mu}^{\nu} = (V - V_{clas})\, \delta_{\mu}^{\nu}$.
- No necessity in any infrared cutoff. All such problems arises only if quantities like $a^3\rho$ are considered which are not normalizable, thus, they may not be considered as probabilities of anything from the mathematical point of view ("unitarity breaking"). Their physical justification is also flawed since it based on the wrong assumption that all Hubble physical volumes ("observers") at given τ are clones of each other while it is not so.
- The accuracy of the stochastic approach is not sufficient for calculating quantities $\sim H^2$ in $<\phi^2>$ and $\sim H^4$ in EMT average values because of the omission of a contribution from the small-scale part (including the conformal anomaly). However, all larger quantities (if exist) can be calculated quantitatively correctly.

Transition to predictions for the post-inflationary evolution

From $\rho(\phi, \tau)$ during inflation to the distribution $w(\tau)$ over the total local duration of inflation:

$$w(\tau) = \lim_{\phi \to \phi_{end}} j = \lim_{\phi \to \phi_{end}} \frac{|V'|}{3H^{n+1}} \rho(\phi, \tau) \, .$$

For the graceful exit to a post-inflationary epoch, the stochastic force should be much less than the classical one during last e-folds of inflation.

QFT of a self-interacting scalar field in the de Sitter background

Starobinsky & Yokoyama (1994).

The equilibrium (static) solution for the 1-point distribution:

$$\rho_{eq}(\phi) = \mathrm{const}\; e^{-2v}\,, \quad v = \frac{4\pi^2 V(\phi)}{3H_0^4}\ .$$

Arbitrary Green functions and n-point distributions can be constructed, too, using solutions of the same Fokker-Planck equation.

$$V(\phi) = V_0 + \frac{1}{2}m^2\phi^2 + \frac{1}{4}\lambda\phi^4\,, \quad 0 < \lambda \ll 1,\ H_0^2 = \frac{8\pi G V_0}{3}\,.$$

Three regimes:

1. Perturbative regime $\sqrt{\lambda}H_0^2 \ll m^2 \ll H_0^2$.

$$< \phi^2 >= \frac{3H_0^4}{8\pi m^2}\left(1 - 3\beta + \frac{429}{16}\beta^2 + \ldots\right), \quad \beta = \frac{3\lambda H_0^4}{8\pi^2 m^4}.$$

Compare to the same result in the one-loop (Gaussian) approximation:

$$< \phi_G^2 >= \frac{3H_0^4}{8\pi m^2}\left(1 - 3\beta + 18\beta^2 + \ldots\right)\,.$$

2. Massless self-interacting regime $|m^2| \ll \sqrt{\lambda} H_0^2$.

$$< \phi^2 >= \sqrt{\frac{3}{2\pi^2}} \frac{\Gamma(0.75)}{\Gamma(0.25)} \frac{H_0^2}{\sqrt{\lambda}} \approx 0.132 \frac{H_0^2}{\sqrt{\lambda}}$$

$$< \phi_G^2 >= \frac{1}{\pi\sqrt{8}} \frac{H_0^2}{\sqrt{\lambda}} \approx 0.113 \frac{H_0^2}{\sqrt{\lambda}}$$

The scale $m^2 \sim \lambda H_0^2$ proposed recently in arXiv:1005.3551 is not critical at all!

3. Symmetry breaking regime $m^2 < 0, \ \ \sqrt{\lambda} H_0^2 \ll |m^2| \ll H_0^2$.

$$< \phi^2 >= \frac{|m^2|}{\lambda} + \frac{3H_0^4}{16\pi^2 |m^2|} + \mathcal{O}\left(e^{-1/(4\beta)}\right)$$

The (modulus of) exponent is the action for the Hawking-Moss instanton.
See also F. Finelli *et al.*, Phys. Rev. D **82**, 064020 (2010).

Probabilities to go to different vacua after inflation

Let inflation may end in two vacua: $\phi = \phi_1$ and $\phi = \phi_2$ with $V(\phi_1) = V(\phi_2) = 0$ (to consider a larger number of post-inflationary vacua, ϕ should have more than one-dimensional internal space).
Boundary conditions at the end of inflation:
$\rho(\phi_1, \tau) = \rho(\phi_2, \tau) = 0$.
Method of calculation (Starobinsky (1984,1986)): consider the quantities

$$Q_m(\phi) = \int_0^\infty \tau^m \rho(\phi, \tau)\, d\tau$$

where $\tau = 0$ corresponds to the local beginning of inflation.
$Q_m(\phi_1) = Q_m(\phi_2) = 0$.

By integrating the Fokker-Planck equation over τ, we get for $m = 0$:

$$Q_0(\phi) = \frac{8\pi^2}{H^{3-n}} \exp\left(\frac{\pi}{GH^2(\phi)}\right) \int_{\phi_1}^{\phi} d\psi \exp\left(-\frac{\pi}{GH^2(\psi)}\right) \times$$

$$\left(C_0 - \int_{\phi_1}^{\psi} \rho_0(\psi_1)\, d\psi_1\right) ,$$

$$C_0 = \frac{\int_{\phi_1}^{\phi_2} d\phi \exp\left(-\frac{\pi}{GH^2(\phi)}\right) \int_{\phi_1}^{\phi} \rho_0(\psi)\, d\psi}{\int_{\phi_1}^{\phi_2} d\phi \exp\left(-\frac{\pi}{GH^2(\phi)}\right)} .$$

$P_1 = C_0$ – the absolute probability to go to the vacuum $\phi = \phi_1$;
$P_2 = 1 - C_0$ – the absolute probability to go to the vacuum $\phi = \phi_2$.
No n dependence in C !

Local duration of inflation

$$Q_1(\phi) = \frac{8\pi^2}{H^{3-n}} \exp\left(\frac{\pi}{GH^2(\phi)}\right) \int_{\phi_1}^{\phi} d\psi \exp\left(-\frac{\pi}{GH^2(\psi)}\right) \times$$

$$\left(C_1 - \int_{\phi_1}^{\psi} Q_0(\psi_1)\, d\psi_1\right),$$

$$C_1 = \frac{\int_{\phi_1}^{\phi_2} d\phi \exp\left(-\frac{\pi}{GH^2(\phi)}\right) \int_{\phi_1}^{\phi} Q_0(\psi)\, d\psi}{\int_{\phi_1}^{\phi_2} d\phi \exp\left(-\frac{\pi}{GH^2(\phi)}\right)}.$$

$$< \tau_1 >= \frac{C_1}{C_0}, \quad < \tau_2 >= \frac{\tilde{C}_1}{1 - C_0},$$

$$< \tau >_{tot} = C_0 < \tau_1 > + (1 - C_0) < \tau_2 > = \int_{\phi_1}^{\phi_2} Q_0(\phi)\, d\phi\,.$$

$\tilde{C}_1$ is C_1 with ϕ_1 and ϕ_2 intechanged.

Choice of an initial condition

- Static solutions – not normalizable in the inflationary (i.e. unstable) case.
- $\rho_0(\phi) = \delta(\phi - \phi_0)$ – why?
- "Eternal inflation as an initial condition": $\rho_0(\phi) \propto \rho_{E_1}(\phi)$ – the wave function of the lowest energy level of the Schrodinger equation arising through the separation of variables in the Fokker-Planck equation ($E_0 = 0$ due to hidden supersymmetry of the former).
  1) Not possible in the continuum spectrum case.
  2) In the discrete spectrum case, generically $E_2 - E_1 \sim E_1$ – not enough time for relaxation.
  As a whole, "eternal" inflation seems not be eternal enough to fix the initial condition uniquely.

However, if inflation had occurred at all, the dependence of predictions on $\rho_0(\phi)$ is comparatively weak: for almost all $\rho_0(\phi)$ except from the HH-like one $\rho_0(\phi) \propto \exp\left(\frac{\pi}{GH^2(\phi)}\right)$, the main contribution comes from the highest maximum of $V(\phi)$ without any necessity of a "tunneling" initial condition.
On the other hand, if $\rho_0(\phi) \propto \exp\left(\frac{\pi}{GH^2(\phi)}\right)$, there is practically no inflation at all, and final probabilities P_1 and P_2 are equal to the initial ones.

Conclusions

- No problems of principle in predicting probability distributions during and after inflation in the original (probability conserving) stochastic approach, once an initial condition $\rho_0(\phi)$ is given. No necessity to refer to other universes outside our light cone.
- No satisfactory principle to fix $\rho_0(\phi)$ uniquely.
- Some dependence on $\rho_0(\phi)$ remains in final answers, so a possibility to get some knowledge on it from observational data does not seem hopeless. However, if inflation had occurred at all, the dependence of predictions on $\rho_0(\phi)$ is weak and mainly produced by the region around the highest maximum of $V(\phi)$. For this, no specific "tunneling" initial condition is needed.