FEATURE

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Research Area : Mathematics

Minimal Model Theory, Derived Categories of Coherent Sheaves, and Mirror Symmetry

Classification Theory of Algebraic Varieties

There is a research field in mathematics called Algebraic Geometry. In algebraic geometry, we study the geometric objects (called algebraic varieties) defined as the solution spaces of polynomial equations. For instance, lines, circles, and parabolas (Figure 1) are algebraic varieties. Since the algebraic varieties are geometric objects, we can study them via geometric intuitions. On the other hand, since they are defined by polynomials, it is also possible to study them via algebraic methods. Also, algebraic geometry is related to several other research fields such as number theory and string theory. For instance, an algebraic variety called an elliptic curve plays an important role in the proof of Fermat's last theorem in number theory, and the three-dimensional Calabi-Yau manifold appears as an extra dimension in string theory. In Japan, the classification theory of algebraic varieties is a central theme in algebraic geometry. The Fields medalists in Japan (Kunihiko Kodaira, Heisuke Hironaka, Shigefumi Mori) all contributed much to the classification theory of algebraic varieties.

The idea of the classification theory of algebraic varieties is, roughly speaking, as follows. First let us consider the simplest case: the one-dimensional case. Although we say one dimensional, the solution spaces of the polynomials are extended to the complex numbers in algebraic geometry, so the real pictures are two-dimensional surfaces (Figure 2). For instance, if we extend the solution spaces to complex numbers, the lines, circles, parabolas become two-dimensional spheres with some punctures. By filling these punctures, (this process is called compactification,) the lines, circles, parabolas all become a two-dimensional sphere. This sphere is called a rational curve, which is the most fundamental one-dimensional algebraic variety. Also an elliptic curve, which is defined by a cubic polynomial, becomes a two-dimensional torus. It is known that all the one-dimensional algebraic varieties are surfaces with some doughnut type holes. The number of the doughnut type holes is called the genus, and the complexity of a one-dimensional algebraic variety depends on its genus: genus zero (sphere), one (elliptic curve), more than one (general type). The idea of the classification theory of one dimensional algebraic varieties is to study their geometric structures once we determine which of the above three types they belong.

Minimal Model Theory for Two-Dimensional Algebraic Varieties

In a higher (more than or equal to two) dimensional case, algebraic varieties are not classified in terms of the numbers of doughnut type holes as in the one-dimensional case. Instead, higher dimensional algebraic varieties are classified in terms of Kodaira dimension, which is different from the usual dimension. The complexity of a higher dimensional algebraic variety depends on its Kodaira dimension, so knowing it

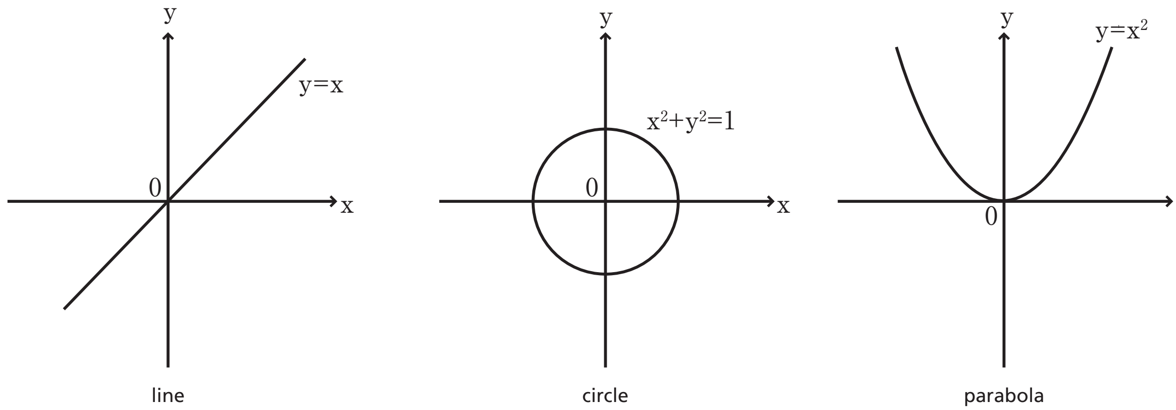

Figure 1

is a key step toward the classification. The global geometric structures of higher dimensional algebraic varieties are complicated, however, and they are not classified as simple as in the one-dimensional case, even if we knew their Kodaira dimension. The reason behind this complexity is that there are extra rational curves on the algebraic varieties that behave badly, so we try to contract an extra rational curve and obtain a new algebraic variety. If we can repeat this process and finally obtain an algebraic variety without such an extra rational curve (called minimal model), then we may try to study its global geometric structure. This is the idea of the higher dimensional classification theory. The above process finding the minimal model, on which there is no extra rational curve, is called the Minimal Model Program (MMP for short).

The MMP for two-dimensional algebraic varieties was completed by an Italian school at the beginning of the 20th century. In this case, there is a further classification of minimal models. For instance, minimal models of the Kodaira dimension zero are classified into four types: K3 surfaces, Enriques surfaces, Abelian surfaces and elliptic surfaces. In each case, there is an interesting geometry behind it. In particular, K3 surfaces are two-dimensional analogue of elliptic curves and three-dimensional Calabi-Yau manifolds, and their geometry is closely related to the lattice theory. Also, since the mirror symmetry of K3 surfaces is described in terms of the lattice, it is actively studied now as a toy model of mirror symmetry.

Minimal Model Theory for Three-Dimensional Algebraic Varieties

As we mentioned above, the two-dimensional minimal model theory was completed in a beautiful way. If we try to construct a similar theory for three-dimensional algebraic varieties, however, we find a serious problem which was not found in the two dimensional case. That is, if we contract an extra rational curve that behaves badly, then the resulting variety may have a singularity. Here we say that an algebraic variety has a singularity if we are not able to find a local coordinate. For instance, we are able to find a (real) two-dimensional coordinate on a one-dimensional algebraic variety since it is a surface with doughnut type holes. Such a coordinate system is not always found in the higher dimensional case. For instance, there is an algebraic variety that looks like a cone, and we cannot find a coordinate system at the vertex. It is difficult to study the geometry of algebraic varieties with singularities, and the three dimensional minimal model theory was not developed for a while.

The above problem was excluded in the 1980's, and the three dimensional minimal model theory was substantially developed. Through the efforts of Shigefumi Mori, Yujiro Kawamata, Vyacheslav Shokurov, and others, a class of singularities (called terminal singularities), which are rather mild and should make the three dimensional MMP work, was introduced and investigated. It is possible that we can contract extra rational curves for varieties with

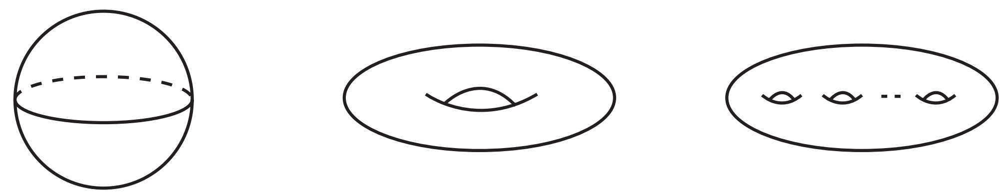

Figure 2

terminal singularities. If the resulting varieties also have at worst terminal singularities, then we can continue the program. Unfortunately, this is not true, since a very bad curve called a flipping curve is contracted to a non-terminal singularity. In this case, the program was shown to continue if we replace the flipping curve to another rational curve. This process is called a flip. The existence of flips was a serious problem, but shown by Shigefumi Mori in 1998, and the three-dimensional minimal model theory was completed.

One of the features of three dimensional minimal model theory is that the resulting minimal models are not unique, but any two of them are connected by a sequence of flops. A flop is very similar to a flip, which replaces a flopping curve that is not so bad as a flipping curve by another rational curve. It was known that flops preserve much geometric information. In the 1990's the ultimate of such a phenomena was found, that is the derived categories of coherent sheaves are equivalent under flops. This was first proved by Alexei Bondal and Dmitri Orlov for particular flops, and later proved by Tom Bridgeland for arbitrary three-dimensional flops.

Derived Categories of Coherent Sheaves

The notion of derived categories of coherent sheaves on algebraic varieties was introduced by Alexander Grothendieck in the 1960's. In order to explain this notion, we first explain coherent sheaves roughly. The notion of coherent sheaves is a generalization of functions on algebraic varieties. For instance, the set of functions on an algebraic variety, which locally written as polynomials, gives a coherent sheaf called a structure sheaf. This is not a unique choice of a coherent sheaf, as structure sheaves on sub algebraic varieties also give more examples of coherent sheaves. There are many coherent sheaves on an algebraic variety, and if we consider each coherent sheaf as an "object" and introduce "morphisms" which relate pairs of coherent sheaves, then we obtain a mathematical system on the set of coherent sheaves. You can imagine this system by considering each coherent sheaf as a point, and a morphism as an arrow between two points corresponding to coherent sheaves. Such a mathematical system, with the notion of objects and morphisms, is called a category.

The category of coherent sheaves is defined as above, but it does not have a good property in some senses. Suppose, for instance, that there is a map between two algebraic varieties and consider the problem associating a coherent sheaf on one of them to one on another variety. There is a naive way of doing this, but it sometimes loses the information of a coherent sheaf. In order to solve this issue, Grothendieck considered complexes of coherent sheaves. Let us explain the complexes of coherent sheaves by comparing them with points and arrows as above. We first put the numbers 1, 2, 3, ⋯ on a finite number of points, and then draw arrows between consecutive numbers from smaller numbers to bigger numbers. We associate coherent sheaves with the numbered points, and morphisms with arrows between consecutive numbers. Such a diagram satisfying a certain property is called a complex of coherent sheaves. The derived category of coherent sheaves is defined to be the category whose objects consist of complexes of coherent sheaves. The morphisms in this category are rather difficult, so we omit the explanation. If we consider the derived category of coherent sheaves, then we can solve the above issue associating a coherent sheaf on a variety to one on another variety. That is, if we associate an object in the derived category of coherent sheaves instead of a coherent sheaf, then we don't lose information.

So far, we have discussed a technical aspect of derived categories. The originally derived category was introduced in order to solve a technical problem, so it was not considered to be related to the geometry of algebraic varieties at the begging of its introduction. Such an idea drastically changed in 1994.

Mirror Symmetry and the Minimal Model Theory

In 1994, at the International Congress of Mathematics held at Zurich, Maxim Kontsevich proposed homological mirror symmetry conjecture. This conjecture predicts equivalence between the derived category of coherent sheaves on an algebraic variety and a certain category (called Fukaya category) determined by a symplectic manifold mirror to it. The idea behind this conjecture is to realize symmetry in string theory by regarding objects in derived categories of coherent sheaves, and Fukaya categories, as D-branes of different types. It was surprising that the derived category of coherent sheaves, which is a rather technical and abstract mathematical notion, was related to string theory. It was also surprising that algebraic geometry and symplectic geometry are predicted to be equivalent, as they seemed to be different geometric theories.

Since the proposal of homological mirror symmetry conjecture, it has been recognized that the derived category of coherent sheaves is an essential mathematical object that realizes symmetry among algebraic varieties. Also through mirror symmetry, several equivalences of derived categories of coherent sheaves on different varieties have been predicted. The derived equivalence under flops is one of them. This is proved for three-dimensional flops, but it is still an open problem in higher-dimensional cases, and a new idea is required.

As a development of the idea of the derived equivalence under flops, it is a natural direction of research to study how derived categories of coherent sheaves behave under steps of MMP. At a special step of MMP, it is observed that the derived category gets smaller by Bondal-Orlov and Yujiro Kawamata. So we expect that the MMP is a program that makes the derived category smaller, and even if the minimal models are not unique, they are equivalent at the level of the derived category. It is a very difficult problem to show this in full generality since we have to deal with singular varieties. If the above idea is realized, however, then it not only provides a new viewpoint of MMP but also several applications are expected.

Now, by a slightly different viewpoint, I am trying to understand each step of MMP as a space of objects (called moduli space) in the derived category on the starting algebraic variety. The keyword is the notion of stability conditions on derived categories introduced by Bridgeland in 2002 inspired by the work of string theory. The results are not satisfactory at this moment, but once this idea is realized, I expect applications to several directions such as quantum invariants and mirror symmetry. In this way, although the minimal model theory was developed in order to classify algebraic varieties, it is now connected with several research fields through string theory and derived categories, and we see a new development of this theory.