Homological Projective Duality and Quantum Gauge Theory

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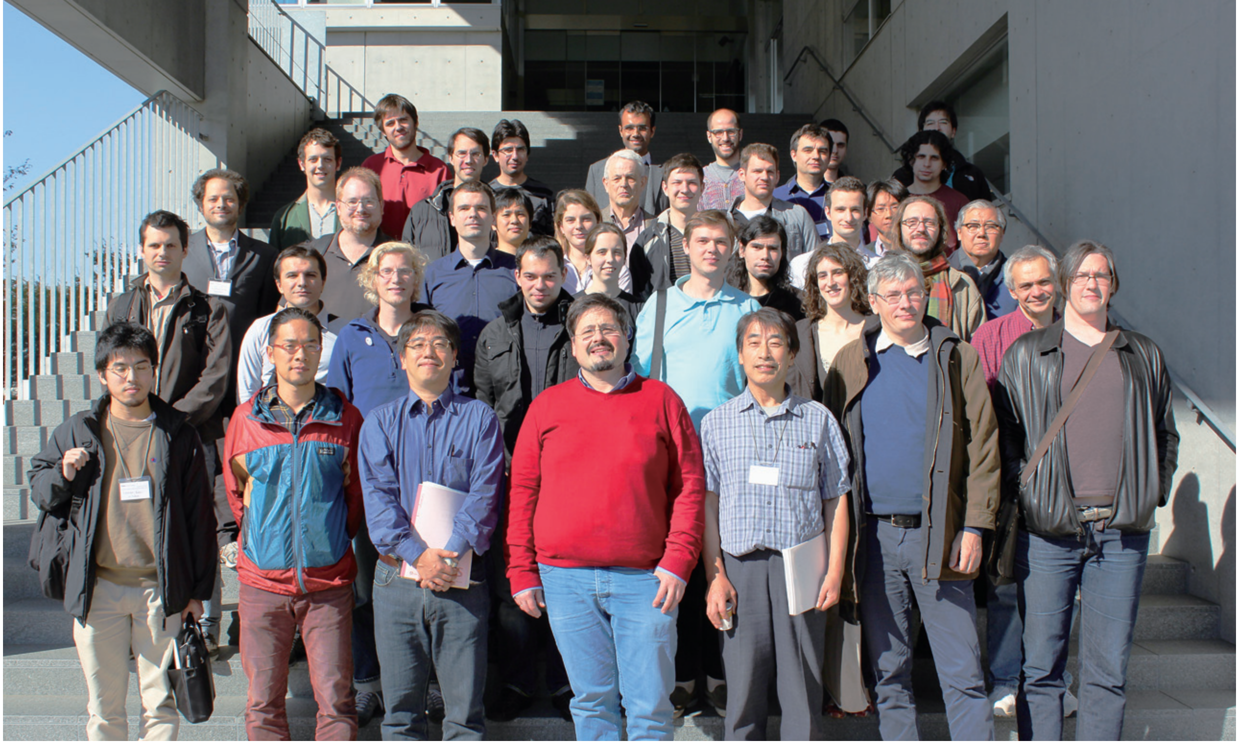
Recently, there has been intriguing and fruitful interaction between one area of mathematics and one area of physics. Physics provides mathematical predictions which mathematicians try to prove, while mathematical results stimulate physicists and sometimes lead to new discoveries. During November 12-16, 2012, this workshop was held in the middle of such interaction, and many of the key players participated.

The area of mathematics is algebraic geometry based on homological algebra and the one of physics is two-dimensional supersymmetric quantum gauge theory. The origin of the interaction goes back to the recognition that the language of category is suited very well to describe a class of objects in string theory, called "D-branes". D-branes are interactions on the worldsheet boundary and they form a type of "category" where open string states play the rôle of "morphisms". Such a category can be the subject of mathematical study, and quite often it is of the type studied earlier, such as "derived category of coherent sheaves". On the other hand, a very powerful method to construct and study quantum field theories on the worldsheet is provided by a class of two-dimensional supersymmetric gauge theories called "linear sigma models". Through this relationship, some facts in two-dimensional gauge theories have some consequences on categories, and some results on categories give some hints toward understanding two-dimensional gauge theories. Below are two examples of such interactions that motivated this workshop.

(1) By a general principle of supersymmetry, the category of D-branes ("B-branes" to be precise) does not change as the parameters (Kähler moduli) are varied. This yields a mathematical prediction that two different categories, corresponding to D-branes at two different regions of the (Kähler) moduli space, must be equivalent. D. Orlov proved examples of such equivalences (2005). He proved that the derived category of coherent sheaves on a projective hypersurface $f=0$ is equivalent to the category of (graded and equivariant) matrix factorizations of f. This proof motivated physicists (M. Herbst, D. Page and myself) to study D-branes in linear sigma models in detail and led them to find the "grade restriction rule", the condition on the gauge charge at the boundary as the theory is deformed from one region to another in the moduli space (2008). This physics result in turn stimulated mathematicians and led them to give a mathematical formulation of the grade restriction rule in much broader contexts (D. Halpern-Leistner, M. Ballard-D. Favero-L. Katzarkov, and W. Donovan and E. Segal, 2012).

(2) In 1998, a Norwegian mathematician E.A. Rødland wrote a mysterious paper which says that two different Calabi-Yau manifolds, call them X and Y, have the same Picard Fuchs equation as their mirror. I learned of this from D. Van Straten in 2003 and tried with D. Tong to explain it by constructing a linear sigma model whose Kähler moduli space has two regions corresponding to X and Y. We managed to do it but it required us to understand



the low energy dynamics of a class of non-Abelian gauge theory in two dimensions (2004-2006). In the meantime, E. Witten pointed out that, if our study goes through, *X* and *Y* must be derived equivalent. I mentioned it to A. Caldararu who was interested in finding examples of birationally inequivalent but derived equivalent varieties. Soon after, Caldararu and L. Borisov came up with a proof of the derived equivalence (2006). It turns out that the equivalence is a particular case of Homological Projective Duality by A. Kuznetsov who also gave a proof. Later, pairs similar to *X* and *Y* were found by A. Caldararu-J. Distler-S. Hellerman-T. Pantev-E. Sharpe (2007) and S. Hosono-H. Takagi (2011). I was naturally interested and tried to explain the new examples using linear sigma model. Again, this required me to understand the low energy dynamics of a different non-Abelian gauge theory. This time, however, it required a different level of understanding and led me to uncover a novel duality in two-dimensional supersymmetric gauge theory (2011), which is similar to Seiberg duality in four-dimensions. The duality provides a unifying scheme to understand many of the known derived equivalences and also produces more examples. This summer in Moscow I discussed with Kuznetsov, and we found that the examples from gauge theory duality and Homological Projective Duality have a significant overlap but there are also some differences.

The workshop was attended by nearly 40 mathematicians and physicists, including many of the persons mentioned above, as well as others who work on different and important aspects of homological algebra, algebraic geometry and supersymmetric gauge theories. It was a great opportunity for all these people to get together in one place and exchange ideas. I very much look forward to seeing where the interaction leads us.

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