FEATURE

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Research Area: Mathematics

From "Local" to "Global" —Beyond the Riemannian Geometry—

From "Local" to "Global"

"Cannot see the forest for the trees." This is a proverb that we will miss the whole picture if focusing only on details. But, is that true? If we observe trees (local properties) in a very clever way (like a detective!) then we might be able to see "something" about the forest (global properties).

In classical mathematics, mainly people studied local phenomena or those objects that are described by local coordinates. In modern mathematics, people's interests have expanded to global objects, and to understand such objects, various ideas and methods have been introduced. Nevertheless, it is in general difficult to understand global properties.

In the study of geometry, the question, "How does local geometric structure affect the global shape?" is a prototype of the following motif:

local structure $\rightsquigarrow$ global nature.

This motif has been a main stream especially in Riemannian geometry since the twentieth century. On the other hand, surprisingly, very little was known about the local-global theory in geometry beyond the Riemannian setting.

Although we just say "local to global," approaches may dramatically change to the types of the local properties. For example, "locally homogeneous" properties are closely related to the theory of Lie groups and number theory. In the case discrete algebraic structure called **discontinuous groups** plays a primary role to control the global structure. In Riemannian geometry, by epoch-making works of Selberg, Weil, Borel, Mostow, Margulis,[1] among others, the study of discrete groups, which ranges from the theory of Lie groups and number theory to differential geometry and topology, developed extensively.

From around the mid-1980s, I began to envisage the possibility of creating the theory of discontinuous groups in the world of pseudo-Riemannian manifolds. Since there is no "natural distance" in such geometries, one has to invent new methods themselves. Although the starting point was solitary, whatever I did was a new development. Since 1990s, a number of mathematicians with different backgrounds have gotten into this theme, and this has brought us to unexpected interactions to other fields of mathematics, such as the (non-commutative) ergodic theory, theory of unitary representations, and differential geometry. In the World Mathematical Year 2000, there was an

[1] A. Selberg was awarded the Fields Medal in 1950, A. Weil was awarded the Kyoto Prize in 1994, A. Borel was awarded the Balzan Prize in 1992, G. Mostow was awarded the Wolf Prize in 2013, and G. Margulis was awarded the Fields Medal and the Wolf Prize in 1978 and 2005, respectively.

occasion that the theme “locally homogeneous spaces (for non-Riemannian geometry)” was highlighted as one of the new challenging problems in mathematics for the twenty-first century ([1]). The study keeps on flourishing even further.

In this article, I would like to deliver the “flavor” of the study of global geometry of locally homogeneous spaces beyond the Riemannian setting and the study of the spectral analysis (global analysis) that we have recently initiated. To do so I try to minimize mathematical terminologies as much as possible; the price to pay is that I may loose certain accuracy of statements.

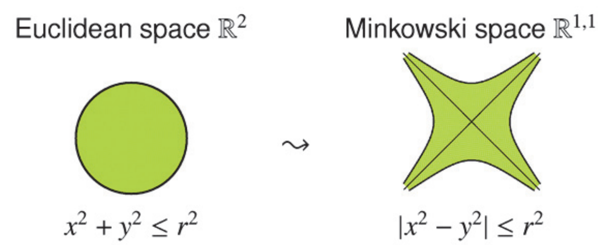
Differential Geometry with Indefinite Signature

Pseudo-Riemannian geometry is a generalization of Riemannian geometry and Lorentzian geometry, which describes the spacetime of the theory of general relativity. On the $p+q$ dimensional Euclidean space, the region defined by the inequality

$$|x_1^2+\cdots+x_p^2-x_{p+1}^2-\cdots-x_{p+q}^2|\leq r^2$$

is called a **pseudosphere**. The figures on the top of the right column illustrate a sphere ($(p,q)=(2,0)$) of the Euclidean space and a pseudosphere ($(p,q)=(1,1)$) of the Minkowski space for the two-dimensional case. For more general situations, by using a non-degenerate quadratic form $Q(x)$ with signature (p,q), we call the region defined by the inequality $|Q(x)|\leq r^2$ a pseudosphere.

In pseudo-Riemannian geometry we deal with the spaces **(pseudo-Riemannian manifolds)**,



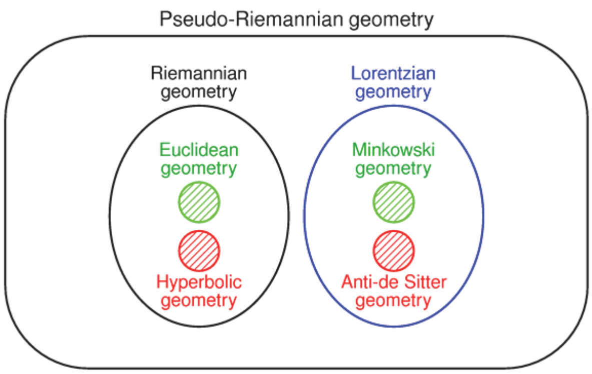
which take pseudospheres as scales at local coordinates (more precisely, at infinitesimal level for each point). When $q=0$ and $q=1$, these spaces are called **Riemanninan manifolds** and **Lorentzian manifolds**, respectively.

To general pseudo-Riemannian manifolds, one can define certain concepts such as the gradient **(grad)**, divergence **(div)**, Laplacian (Δ=div $\circ$ grad), and curvature. Moreover, in the case of Riemannian manifolds, that is when $q=0$, since the quadratic form $Q(x)$ is positive-definite, one can also define the distance between two points by integrating (infinitesimal) scales. On the other hand, in the case of pseudo-Riemannian manifolds with indefinite signature $p,q\geq 1$, there is no reasonable way to define the distance: the “intrinsic distance” does not exist.

Is a Wayfarer Coming Back?

The Earth is round. A wayfarer traveling towards the west would eventually come back from the east. By the way, if the wayfarer does not know any global facts on the Earth, such as the shape or size, then is there any way for them to know whether they will come back to the starting point?

In mathematics we describe by the quantity



called **curvature** how spaces are curved at the infinitesimal level. The concept of curvature traces back to Carl Friedrich Gauss in the nineteenth century, who also contributed to geodesy during his long profession as the director of Göttingen Observatory. For two-dimensional curved surfaces there are the following visible relationship between the curvature and local shapes:

positive curvature
$\Leftrightarrow$ locally concave up or concave down,
negative curvature
$\Leftrightarrow$ locally saddle-shaped.

In higher dimensional spaces there are three kinds of curvature, namely, sectional curvature, Ricci curvature, and scalar curvature, where sectional (resp. scalar) curvature contains the most (resp. least) information.

How does curvature (local geometric structure) affect the **global shape?** The classical Myers theorem states: "If the Ricci curvature is greater than 1 then the distance between any two points is less than π (for any dimension)." This is a local-global theory. So, from the curvature of the surface of the Earth (local information), one can obtain global information, namely, the diameter of the Earth. The claim of the theorem, "If a surface is locally convex everywhere then as a whole space it is closed like a sphere," agrees with our experience in our daily life. However, when the curvature is negative or in pseudo-Riemannian geometry, can a wayfarer who is going forward in a uniformly curved space come back to the starting point? From the next section we shall enter some strange world that cannot be understood by our "daily life."

Uniformly Curved Geometry

Pseudo-Riemannian manifolds of constant curvature are called **space forms**, and play an important role in differential geometry. The constant-curvature property is one of the special examples of local homogeneity. Due to their high symmetry, space forms are interacted with various fields of mathematics.

In Riemannian geometry, the sphere, Euclidean space, and hyperbolic space are the space forms of positive, zero, and negative curvature, respectively. The hyperbolic space is also known as to have played a historical role in the early nineteenth century on a discovery of geometry that does not hold the parallel postulate on Euclidean geometry. In the three-dimensional case the theory of hyperbolic manifolds, which are Riemannian manifolds of sectional curvature -1, is in fact equivalent to that of the Kleinian group. Hyperbolic geometry is an active research field.

In Lorentzian geometry, which is the simplest geometry beyond the Riemannian setting, the **de Sitter manifold**, Minkowski space, and **anti-de Sitter manifold** are space forms of positive, zero, and negative curvature, respectively.

Is the "Universe" Closed?

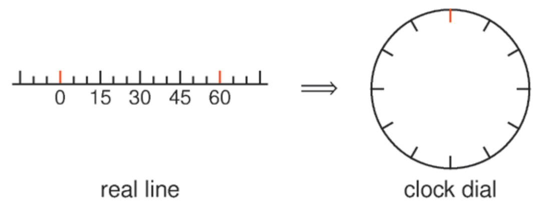
Does there exist any space that curves locally the same everywhere and also that is closed globally?[2] In space forms of positive curvature, which generalizes the concept of "being concave up," the following theorem holds.

Theorem 1. Space forms of positive curvature are
(1) *always* closed (Riemannian geometry).
(2) *never* closed (Lorentzian geometry).

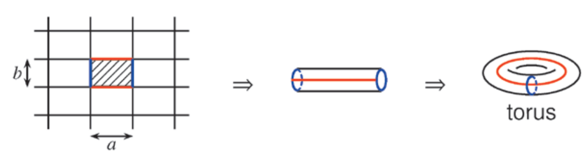
Theorem 1 (2) is called the **Calabi–Markus phenomenon**, named after the two mathematicians E. Calabi and L. Markus, who discovered this surprising fact ([2]).

On the other hand, the standard model of space forms of negative curvature expands infinitely and so it is never closed. For next, in order to find a *closed* "universe" having the same local geometric structure, I would like to illustrate an idea to "fold" the open "universe" by one-dimensional and two-dimensional Euclidean spaces as elementary as possible.

The real line is obviously not closed, while a clock dial, on which the long hand goes around in every sixty minute (period), is closed. No matter which ones we use, we record the same time locally. As in this example, it is possible that two spaces are locally the same but different globally.



Let us observe a similar process for the two-dimensional Euclidean space $\mathbb{R}^2$. If there is a period both in the vertical direction and in the horizontal direction then one can tile the plane by rectangles, which represent the period. Moreover, if identifying the edges of the rectangles with one period then, by gluing the edges, one can obtain a closed shape called a torus (a surface of a doughnut).

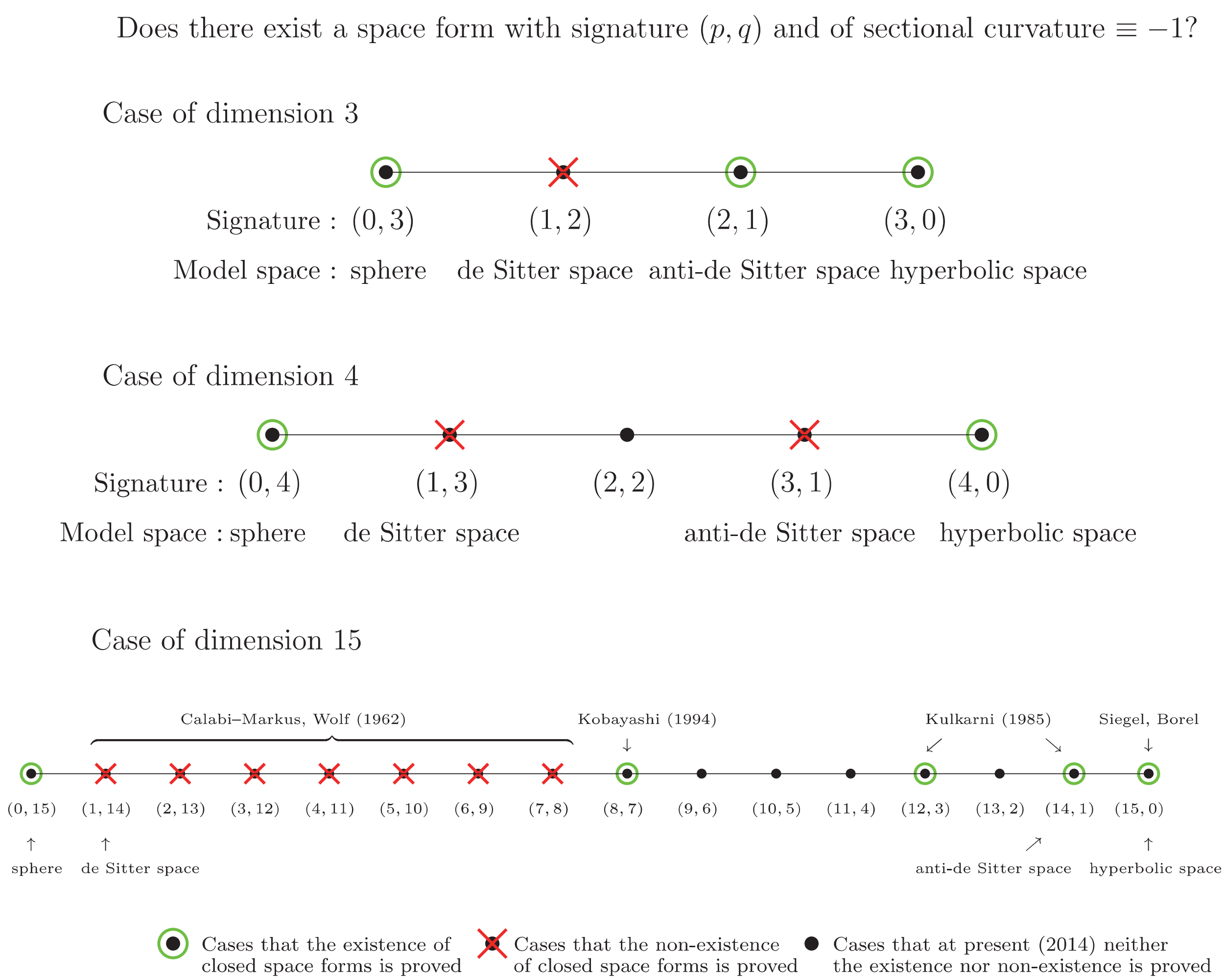


The important principles behind these elementary examples may be formalised as

A. the algebraic structure that represent the period (**discontinuous group** $\mathbb{Z}^2$),
B. tiling (by rectangles).

Once finding the principles like A or B, with keeping local structures, one may expect to produce spaces of different global shapes; nontheless, it is in general far more difficult than the above examples,

[2] In mathematics, it is stated as, "Does there exist a compact pseudo-Riemannian manifold of constant curvature?"



as some non-commutative structure emerges.

On the existence problem of closed space forms of negative curvature, it is known that the following theorem holds.

Theorem 2. Closed space forms of negative curvature exist
(1) *for all* dimensions (Riemannian geometry).
(2) *only for odd* dimensions (Lorentzian geometry).

A proof for Theorem 2 is in fact given by the algebraic idea of A to use matrices with integral entries (arithmetic lattice). Moreover, in the case of Theorem 2 (1), Mostow, Vinberg, Gromov, and Piateski-Shapiro gave another construction method by a use of the geometric idea of B (non-arithmetic lattice) for space forms of negative curvature (hyperbolic manifolds). On the other hand, in Lorentzian geometry, as Theorem 2 (2) exhibits, there is a difference between the odd dimensional case and even dimensional case. This difference can be explained by a topological method, with which one can give a mathematical proof for the theorem, "There always exists the whorl of hair on our head."

Theorems 1 and 2 both claim that there exists a significant difference between Riemannian geometry and Lorentzian geometry on the motif, "local structure $\rightsquigarrow$ global nature." What about pseudo-Riemannian geometry with more general signature (p,q) $(p \geq q \geq 2)$? In the case of positive curvature it is known that closed space forms do not exist (a generalization of the Calabi–Markus phenomenon). On the other hand, in the case of negative curvature, the existence problem, namely, for which integers p,q closed space forms exist, has not been completely solved. As partial results, it has been proved that there exists a closed space form of negative curvature in the case of dimension 7 with signature (4,3) and dimension 15 with signature (8,7), for instance.

The table in the previous page illustrates the current states of the arts on the existence problem of closed space forms, by taking dimensions to be 3, 4, and 15 as examples ([4]).

Rigidity and Deformation

A **rigidity theorem** represents a property that the same type of local geometric structure can be equipped uniquely in a given global geometric structure. Conversely, if there is some freedom for the local structure to be equipped then the freedom itself will become a subject of study (**deformation theory**).

In Riemannian geometry rigidity theorem has been found in various formulations. As an exceptional example, it is known that there are continuously many distinct hyperbolic structures (Riemannian structures of curvature –1) on a closed two-dimensional surface. The parameter space (up to certain equivalence) is called the **Teichmüller space**, which is connected with different disciplines of mathematics from complex analysis and hyperbolic geometry to string theory in theoretical physics. In the two-dimensional hyperbolic geometry since the discontinuous groups that control the global shapes are discrete subgroups of $SL(2,\mathbb{R})$ (Fuchsian groups), the Teichmüller space can be thought of as the deformation space of the Fuchsian groups.

In pseudo-Riemannian geometry with indefinite signature, we find that the geometric structure tends to be less rigid and more “flexible.” I believe that the deformation theory of discontinuous groups in the pseudo-Riemannian setting has a good potential to yield fruitful theories in future ([1]).

Shorter Strings Produce a Higher Pitch than Longer Strings —Spectral Geometry—

One of outstanding perspectives in modern mathematics is that the study of geometry (“shapes”) is equivalent to studying the functions (“residents”) on the geometry. This point of view has brought us a great success in algebraic geometry and some other fields of mathematics.

As we mentioned above, there are some strange geometries, which are locally the same but which may be globally different. Now let us apply the above perspective to the strange geometries and consider the “residents” on them in the pseudo-Riemannian setting.

For a string with constant tension, the longer the string becomes, the lower the pitch the string makes is. As an elementary model, we consider the eigenfunctions for the Laplacian with period L on the real line. We then see that the larger the period becomes, the smaller the eigenvalues are. For example, trigonometric function $f(x) = \sin(2\pi x/L)$ has period L and satisfies the differential equation

$$-\frac{d^2}{dx^2} f(x) = \frac{4\pi^2}{L^2} f(x).$$

Therefore, if the period L is getting larger then the eigenvalue $4\pi^2/L^2$ is becoming smaller.

In Riemannian geometry a similar phenomenon holds. Suppose that we list all the eigenvalues for the Laplacian on a two-dimensional closed hyperbolic surface in increasing order:

$$0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots$$

In this case it is known that if we continuously deform the Riemannian metric with keeping the hyperbolic structure, then all the eigenvalues λ_k $(k = 1,2,\ldots)$ will vary ([5]). In other words, when eigenvalues are regarded as functions on the Teichmüller space, they cannot be constant.

Functions on the closed hyperbolic surface are identified with periodic functions on the upper-half plane, where the period is given by the Fuchsian group (disconnected group). Therefore the above example illustrates how eigenvalues vary according to the deformation of the Fuchsian group.

Universal Sounds Exist for "Music Instruments" in the Anti-de Sitter Universe

Beyond Riemannian geometry, the global analysis on locally homogeneous spaces is still veiled in mystery and is an unexplored area. It is the study of "residents" on geometries. As a first step of its exploration, we have discovered the following a bit surprising phenomenon.

Theorem 3. On any three-dimensional closed anti-de Sitter manifold, there exist infinitely many stable eigenvalues for the Laplacian.

This theorem captures the phenomenon which contradicts our "common sense" for music instruments: "The longer the string becomes, the lower the pitch the string makes is." On the other hand, as in the two-dimensional closed hyperbolic surface described in the previous section, there still exist infinitely many eigenvalues of the Laplacian which vary according to the period in the "standard" three-dimensional closed anti-de Sitter manifold.

If we consider anti-de Sitter manifolds as music instruments then they will be such instruments which have infinitely many "universal sounds"; no matter how long or short the strings become, their pitch do not vary. At the same time the music instruments also have infinitely many sounds that DO vary (as usual)!

The new phenomenon that was discovered by Theorem 3 occurs in higher dimensional cases and also in some locally symmetric spaces with indefinite Kähler metric. The proof of the general theory is given in our recent article of 140 pages ([6]). The main tools are

- partial differential equations,
- integral geometry,
- non-commutative harmonic analysis, and

・quantitative estimates of proper actions.

In a subsequent paper we plan to make a bridge between infinite-dimensional representation theory and the global analysis of locally homogeneous spaces, by applying to

・the theory of **branching laws** for breaking symmetries of infinite dimensional spaces.

Over the half-century the global analysis on the locally homogeneous space $\Gamma \backslash G / H$ has been deeply developed as an important branch of mathematics in the following special cases.

・The case that H is compact (G / H is a Riemannian symmetric space): the theory of **automorphic forms** in number theory (Γ is an arithmetic subgroup.)

・The case that Γ is a finite group consisting only of the identity element: **non-commutative harmonic analysis**, developed by I.M. Gelfand, Harish-Chandra, and T. Oshima, among others.

In contrast to the cases given above, the geometry of pseudo-Riemannian locally homogeneous spaces that we have discussed in this article is more general, *i.e.* the local geometric structure is no more Riemannian (that is, H is not compact) and global geometric structure is given by an infinite discontinuous group Γ. This generalization has opened the door of a new area of geometry beyond the Riemannian setting. What mysteries are in the spectral analysis for such a geometry, which studies the "residents" in the strange universe? I believe that there is an interesting future of the study, which we seem to start glimpsing in distance.

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