Special Contribution

Moonshiney Conference

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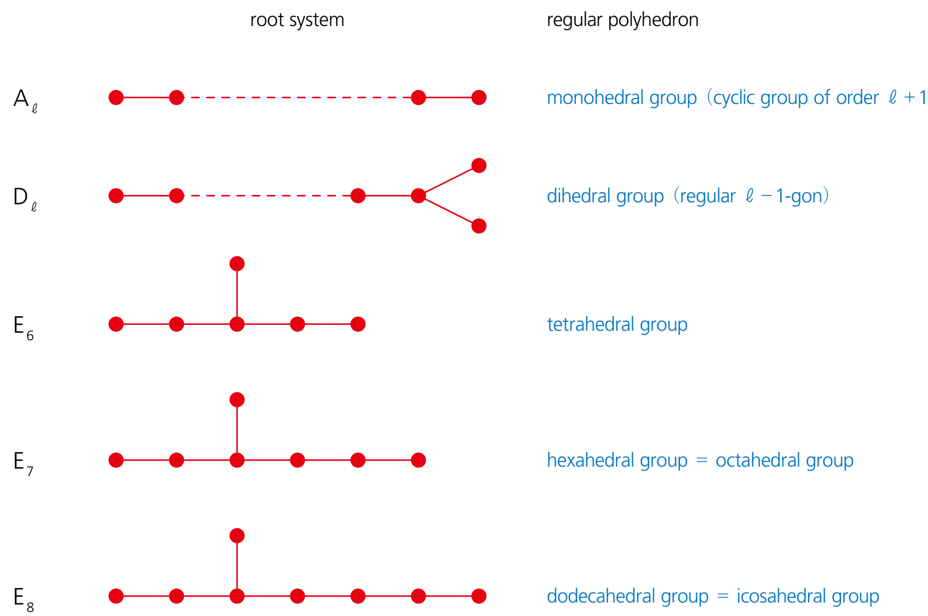
Symmetry is one of clues that human beings use to recognize the world. For instance, the human face has a left-right symmetry, and a round flower has a rotation symmetry (when viewed from above). Particularly well-known is the symmetry of regular polyhedra, which are shown on page 26. Plato is said to have recognized the harmony of the universe in this symmetry, and so he ordered diagrams of regular polyhedra to be displayed at his school, Academeia. Surprisingly, human beings appear to have first recognized this symmetry much earlier than Plato's time - stones carved into the shape of regular polyhedra have been discovered amongst New Stone Age relics in Scotland.

However, it was not until much later, in the early 1800s, that symmetry was formulated precisely in a mathematical sense with the notion of the group (a system of operators satisfying the associative law and containing the unit element, with every element having an inverse). Évariste Galois, who died at the tender age of 20 in May 1832 in a duel, devised the notion of a normal subgroup of a group to completely settle the solvability of algebraic equations, which was discovered in letters he wrote the night before the duel. In the letter, Galois also suggested application of the notion of groups to permutations of Abelian integrals.

Later, from around 1870, Felix Klein, Sophus Lie, and other mathematicians began to apply the study of the structure of groups, which had been initiated by Galois, to geometry. In particular, it is known that there is a one-to-one correspondence between the table of classification of simple Lie groups (to be precise, classes which are called simply-laced types), first performed in 1888 by Wilhelm Killing, Élie Cartan, and other mathematicians, and that of regular polyhedra. (There are several known correspondence methods. One method in particular, the method using the characters of regular polyhedral groups, is called the McKay Correspondence.) The Table of Correspondence Methods and the Dynkin Diagrams, which are used to classify simple Lie groups, are shown below.

Independent of this, many mathematicians studied simple groups (groups which are no longer decomposed by their normal subgroups). It is said that the classification of simple groups consisting of finitely many elements was finally completed in 2004. There are 26 groups arising sporadically in the classification table apart from some infinite sequences of groups, including those described by simple Lie groups. The largest group amongst these sporadic groups is called the Monster (named by John Conway). Conway, Simon Norton, Robert Griess and others pointed out that the Monster, if it should exist, can be realized as the symmetry of a 196883-dimensional space. Finally Griess constructed a 196883-dimensional non-associative commutative



algebra, and this confirmed the existence of the Monster as the symmetry (the automorphism group) of the algebra.

John McKay pointed out that the number at the focus of interest, 196883+1, arises as a coefficient in the Fourier expansion of the elliptic modular function j, $j(z)=q^{-1}+744+196884q+$...... This famous discovery is known as McKay's Observation. As the theory of elliptic functions, stemming from Leonhard Euler's work in the 1740s, had developed as one of the most historic and important fields in mathematics, a huge sensation was created by the suggestion that elliptic functions are (likely to be) closely related to the Monster group. People therefore came to take notice of Fourier coefficients of modular functions, and a series of amazing discoveries called Monstrous Moonshine followed immediately. These include the correspondence between the conjugacy classes of the Monster and the elliptic modular functions and groups (the Conway-Norton Correspondence, 1979) and an explanation of this fact by a vertex operator algebra (Richard Borcherds).

The Moonshiney Conference held at IPMU in Kashiwa, Japan, was lively and exciting, with attendance by mathematicians involved in Monstrous Moonshine - John McKay, John Conway, Robert Griess, and others. I was particularly impressed with the lectures by McKay and John Duncan. The former told us about a new movement in the study of Monstrous Moonshine, and the latter sought to explain "another McKay Correspondence" (conjugacy classes of products of Fischer involutions in the Monster from the E_8 Dynkin Diagram) by using the corresponding elliptic modular group. The evolutionary history of symmetry started with the symmetries of regular polyhedra known by human beings from ancient times, went through the symmetry of type E_8, and has now come to the symmetry of the newcomer, the Monster. I dream that this evolution might extend to symmetries arising in the deepest parts of the universe in the future.

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