FEATURE

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Research Area : Mathematics

Braids and 3-Dimensional Geometry Interacting with Physics

Interactions between geometry and physics

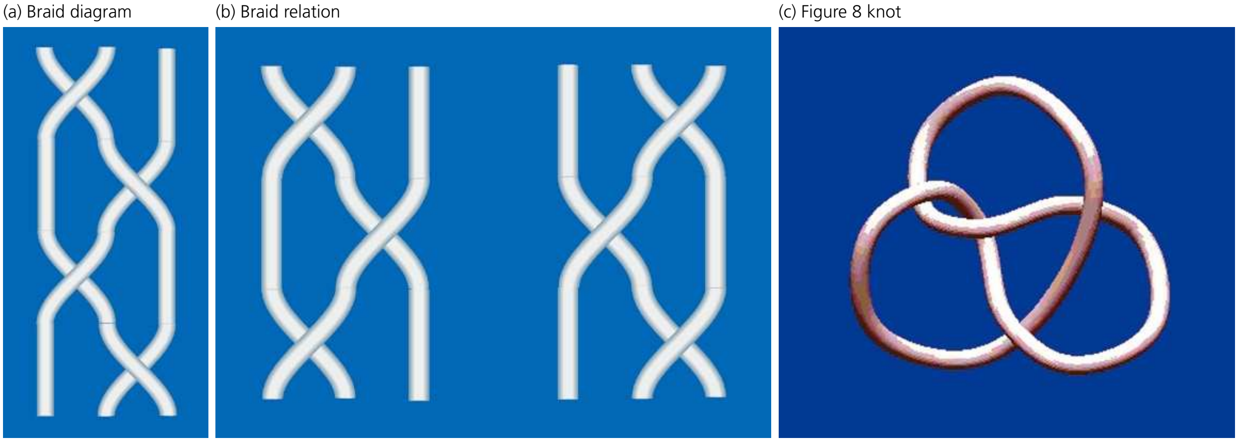
Geometry and physics have been developed with a strong influence on each other. It is well known that the differential geometry developed by Gauss and Riemann from the middle of the 19th century became the basis of Einstein's general relativity. The classical mechanics founded by Newton, through to the formulation of analytic mechanics by Lagrange and Hamilton, yields an important area of contemporary geometry called symplectic geometry. As we see in these examples the interactions between geometry and physics occur sometimes in an unexpected way. One of the most remarkable features of the new developments in the last few decades is that the quantum field theory ties up with the deep properties of topology. In this article we shall take a glance at recent trends in these research areas, focusing on the theory of braids and 3-dimensional geometry.

Braids and the Dirac spinor

A braid is represented by a diagram with a bunch of vertical strands linked to each other as shown in Figure 1 (a). It is essential to distinguish over crossings and under crossings. We define the product of two braids by composing them in the vertical way. The braids shown in Figure 1 (b) are obtained one from the other by moving strands fixing the endpoints.

Figure 1

(a) Braid diagram (b) Braid relation (c) Figure 8 knot



In general we shall identify braids obtained by such a continuous deformation of strands. Figure 1 (b) shows the most fundamental relation among braids. The notion of braids was defined by Artin in the middle of 1920's.

The beginning of the 20th century was the era of the establishment of quantum mechanics as well. Dirac was a physicist who noticed the importance of the notion of braids in an early stage. He effectively exploited braids for explaining the notion of spinors in quantum mechanics. The spinor is a multi-valued quantity so that it changes the sign by the rotation of 360 degrees and becomes identical by the rotation of 720 degrees. Let us perform a simple experiment using a belt as shown in Figure 2. We fix one end point and make two full twists by rotating the other end by 720 degrees. Then we move the end we have twisted as shown in Figure 2 and finally obtain a belt without twists. One full twist cannot be resolved in this way. This shows in terms of braids that a braid for the rotation of 360 degrees on the sphere cannot be trivialized by a continuous deformation, while a braid for the rotation of 720 degree can be trivialized, which is related to the existence of the spinor.

From braids to infinite-dimensional geometry

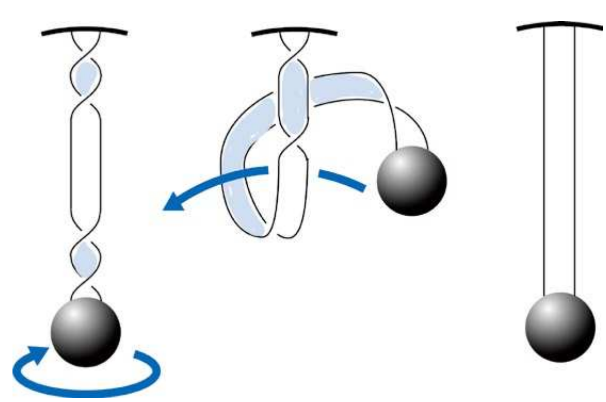
On the other hand, braids play an important role in topology. By closing braids we obtain knots or links. A knot is an embedded closed curve in the 3-dimensional space and a link is a disjoint union of embedded closed curves. For example, closing the braid shown in Figure 1 (a) we obtain the knot in Figure 1 (c). Knots are remarkably complicated objects and, even with all the sophisticated techniques of contemporary topology, they have resisted a definitive treatment. We could say that difficulty in 3-dimensional topology is concentrated on the complexity of knots.

In the middle of 1980's, Jones invented a significantly novel technique in knot theory. This was the discovery of the Jones polynomial derived from the theory of operator algebras. Representing braids by linear operators Jones extracted new topological invariants for knots and links. This invariant was completely different from known invariants based on classical techniques in topology such as the Alexander polynomial.

A few year later Witten proposed a formulation of the Jones polynomial by means of the quantum field theory. He defined it as the partition function of the 3-dimensional Chern-Simons gauge theory. This is given as a certain average of infinitely many quantities and its topological invariance manifests in its form as far as it is mathematically well defined. Here the topological invariance means that the defined quantity does not depend on information derived from metrics such as length and angle etc., and is invariant under continuous deformation.

The Jones polynomial can also be interpreted as

Figure 2 Belt trick



the partition function in statistical mechanics. The braid relation shown in Figure 1 (b) is written as ABA=BAB and is called the Yang-Baxter equation. It is an integrability condition in statistical mechanics and the algebraic structure behind it was formulated as the notion of quantum groups by Drinfel'd and Jimbo. Moreover, the theory of Witten formulated as 2+1 dimensional topological quantum field theory reveals a relationship with the 2-dimensional conformal field theory. The conformal field theory was initiated by physicists Belavin, Polyakov and Zamolodchikov as the theory for infinite dimensional symmetries in critical phenomena in statistical mechanics. Later on Tsuchiya and his collaborators established its mathematical foundation based on the notion of infinite dimensional Lie algebras. Chern-Simons theory, conformal field theory and quantum groups are intricately related with the key word of the Jones polynomial. Researchers in mathematics at IPMU have made a considerable contribution to the development of these areas.

Geometrization of spaces

Given a local geometric structure of a space, determining its possible global geometric structure is an important problem. This research area has

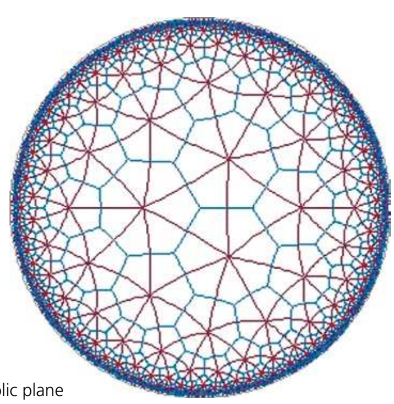
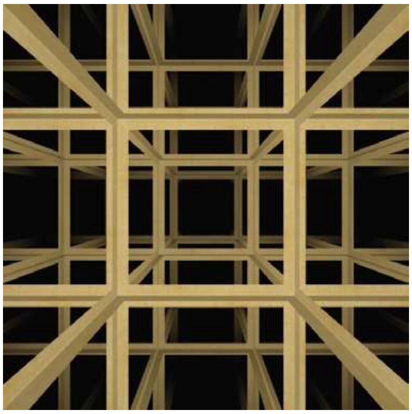


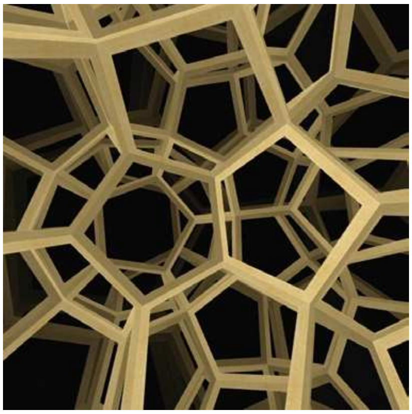
Figure 3
Tessellation of the hyperbolic plane

been developed as one of the principal trends in geometry since the 20th century. For example, a closed oriented surface with a metric so that it is locally isometric to the Euclidean plane is known to be a torus. Closed oriented surfaces are topologically classified into a sphere, a torus and surfaces of genus greater than 1, which are obtained as connected sum of tori. The number of tori appearing in the connected sum is called the genus. We shall say that a torus has a geometric structure modeled on the Euclidean plane. A surface of genus greater than 1 has a geometric structure modeled on the hyperbolic plane as shown in Figure 3. Figure 3 shows a tessellation of the hyperbolic plane by regular triangles. We consider the metric inside the disc so that all the triangles are isometric to each other. This model of hyperbolic geometry is called the Poincaré disc. In this way we see that there are 3 kinds of geometric structures for closed surfaces, spherical geometry, Euclidean geometry and hyperbolic geometry. The curvature is a positive constant for spherical geometry, is zero for Euclidean geometry and is a negative constant for hyperbolic geometry. The angle sum of a triangle is more than 180 degrees in the spherical case and is less than 180 degrees in the hyperbolic case.

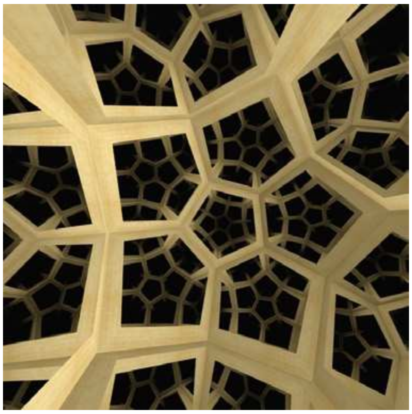
The next issue is to investigate possible geometric structures for 3-dimensional spaces. This is the problem of geometrization of spaces. As in the 2-dimensional case there are 3 kinds of models of constant curvature, 3-dimensional spheres (positive constant curvature), Euclidean space (zero curvature) and 3-dimensional hyperbolic space (negative constant curvature). There are miscellaneous models constructed from 1-dimensional and 2-dimensional geometric structures and in total we have 8 kinds of models for 3-dimensional geometric structures. We



(a) Flat space



(b) Positively curved space



(c) Negatively curved space

Figure 4 These picture are drawn by the software [3] developed by Weeks.

show in Figure 4 tessellations by regular polyhedra of the 3 kinds of model spaces of constant curvature. An approach to 3-dimensional topology by means of these geometric structures was initiated by Thurston in the 1980's with a huge impact on the research in topology.

It is known that the complement of the figure 8 knot shown in Figure 1 (c) is equipped with a complete hyperbolic metric. It was shown by Thurston that with finitely many families of exceptions the complements of allmost all knots have such hyperbolic structures. This led to important progress in the theory of knots and braids. The geometrization for general 3-dimensional spaces was conjectured by Thurston and settled by Perelman a few years ago. Perelman's method exploits the asymptotic behavior of the solutions of the differential equation for the metrics called the Ricci flow. In this approach techniques derived from physics such as renormalization play an essential role in controlling divergent solutions. This is a new interaction between geometry and physics.

References:
[1] M. Atiyah, Geometry and Physics of Knots, Cambridge University Press, 1990.
[2] T. Kohno Conformal Field Theory and Topology, Translations of Mathematical Monographs, Amer. Math. Soc. 2002.
[3] J.R. Weeks, http://www.geometrygames.org/CurvedSpaces/