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Research Area : Mathematics

Jugendtraum of a mathematician

§1 Kronecker's Jugendtraum

There is a phrase "Kronecker's Jugendtraum (dream of youth)" in mathematics. Leopold Kronecker was a German mathematician who worked in the latter half of the 19th century. He obtained his degree at the University of Berlin in 1845 when he was 22 years old, and after that, he successfully managed a bank and a farm left by his deceased uncle. When he was around 30, he came back to mathematics with the study of algebraic equations because he could not give up his love for mathematics. Kronecker's Jugendtraum refers to a series of conjectures in mathematics he had in those days — maybe more vague dreams of his, rather than conjectures — on subjects where the theories of algebraic equations and of elliptic functions intersect exquisitely. In the present note, I will explain the dream itself and then how it is connected with my dream of the present time.

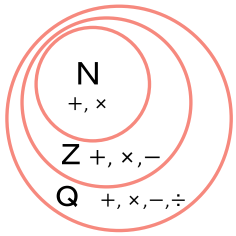
§2 Natural numbers **N**, integers **Z** and rational numbers **Q**

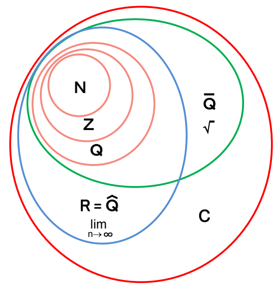
Let us review systems of numbers for explaining Kronecker's dream. Some technical terms and symbols used in mathematics will appear in the sequel and I will give some comments on them, but please skip them untill §9 and §10 if you don't understand them.

A number which appears when we count things as one, two, three, ... is called *a natural number.* The collection of all natural numbers is denoted by **N**. When we want to prove a statement which holds for all natural numbers, we use mathematical induction as we learn in high school. It can be proved by using induction that we can define addition and multiplication for elements of **N** (that is, natural numbers) and obtain again an element of **N** as a result. But we cannot carry out subtraction in it. For example, $2-3$ is not a natural number anymore. Subtraction is defined for the system of numbers ..., -3, -2, -1, 0, 1, 2, 3, ... We call such a number an *integer* and denote by **Z** the collection of integers. For **Z**, we have addition, subtraction, and multiplication, but still cannot carry out division. For example, $-2/3$ is not an integer. A number which is expressed as a ratio p/q of two integers ($q \neq 0$) is called a *rational number* (in particular an integer is a rational number) and the collection of them is denoted by **Q**. Rational numbers form a system of numbers for which we have addition, subtraction, multiplication, and division.[*1] Such a system of numbers is called a *field* in mathematics.

We ask whether we can measure the universe by rational numbers. The answer is "no" since they still miss two type of numbers: (1) solutions of algebraic equations, (2) limits of sequences. In the follwoing §3 and §4, we consider two extensions $\bar{\mathbf{Q}}$ and $\hat{\mathbf{Q}}$

of $\mathbf{Q}$, and in §6, both extensions are unified in the complex number field $\mathbf{C}$.





§3 Algebraic numbers $\overline{\mathbf{Q}}$

It was already noticed by ancient Greeks that one cannot "measure the world" only by rational numbers. For example, the length of the hypotenuse of a right-angled isosceles triangle with the short edges of length 1 is denoted by $\sqrt{2}$ (Fig. 1) and Greeks knew that it is not a rational number. If we express $\sqrt{2}$ by the symbol x, then it satisfies the equation $x^2-2=0$. In general, a polynomial equality including an unknown number x such as $a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n = 0$ ($a_0 \neq 0$, a_1, ..., a_n are known numbers called coeffcients) is called an algebraic equation. We call a number x an *algebraic number* if it satisfies an algebraic equation with rational number coeffcients. The collection of algebraic numbers, including rational numbers, is denoted by $\overline{\mathbf{Q}}$. It is a field since it admits addition, subtraction, multiplication, and division. Moreover we can prove that solutions of algebraic equations whose coeffcients are algebraic numbers are again algebraic numbers. Referring to this property, we say $\overline{\mathbf{Q}}$ is algebraically closed. This $\overline{\mathbf{Q}}$ is an extremely exquisite, and charming system of numbers, but we are far from complete understanding of it despite the full power of modern mathematics. Is $\overline{\mathbf{Q}}$ sufficient to mesure the world? Before answering the question, let us consider another extension of systems of numbers in the next section.

§4 Real numbers $\mathbf{R}$

Analysis was sarted in modern Europe by Newton (1642-1723) and Leibniz (1646-1716) and followed by Bernoulli and Euler. It introduced the concept of approximations of unknown numbers or functions by sequences of known numbers or functions.[*2] At nearly the same time in modern Japanes mathematics (called *Wasan*), started by Seki Takakazu (1642(?)-1708) and developed by his student Takebe Katahiro (1664-1739), approximations of certain inverse trigonometric functions by a power series and that of π by series by rational numbers were also studied. Takebe wrote "I am not so pure as Seki, so could not capture objects at once algebraically. Instead, I have done long complicated calculations." We see that Takebe moved beyond the algebraic world, an area of expertise of his master Seki, and understood numbers and functions which one can reach only by analysis (or series). Nowadays, a number which "can be approximated as precisely as required by rational numbers" is called a *real number* and the whole of them is denoted by $\mathbf{R}$.[*3] A number which has an infinite decimal representation (e.g. π =3.141592 ...) is a real number and the inverse is also true. Thus, numbers, which we learn in school, are real numbers. Japanese mathematicians of the time had high ability to calculate such aproximations by using abacuses, and competed with each other in their skills. However I don't know to what extent they were conscious about the logical contradiction that one cannot reach real numbers in general without infinite approximations, while the size of an abacus is finite (even nowadays, we meet the same problem, when we handle real numbers by computer). In Japan, we missed the tradition of Euclid. Some people, old Archimedes in Greek,

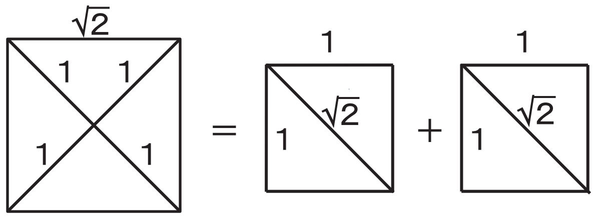


Figure 1: We consider a square whose diagonals have the length 2. Then its area is equal to 2, since we can decompose and rearrange the square into two squares whose side lengths are equal to 1. Therefore, the side length of the original square is equal to $\sqrt{2}$.

Cauchy in France, Dedekind in German and his contemporary Cantor, tried to clarify the meaning of "can be approximated as precisely as required" and now the system of real numbers **R** is usually described according to their work. However, because of an embarrassing problem found by Cantor,[*4] understanding of **R** involves another kind of hard problem than that of $\bar{\mathbf{Q}}$.

§5 Algebraic numbers versus real numbers

Incidentally, I think many mathematicians concern either the understanding of **R** or that of $\bar{\mathbf{Q}}$ and have their opinions. Some years ago, I talked with Deligne, a great mathematician in this age, at a conference about the completeness of real numbers. I was deeply impressed, when I heard him regretfully saying "Real numbers are diffcult. We are far from understanding them". Actually, $\bar{\mathbf{Q}}$ has a clue called the absolute Galois group, which aids our understanding of it,[*5] while **R** consists of all convergent series, which offers little clue for capturing its elements (in spite deep theory of approximations of irrational numbers by rational numbers).

§6 Marvelous complex nubers C

A complex number z is a number expressed as $z = a + bi$, using two real numbers a and b where the symbol i (called the imaginary unit) satisfies the relation $i^2 = -1$. The whole $\mathbf{R} + \mathbf{R}i$ of all *complex numbers*, denoted by **C** and called complex number field, carries the both properties: i) algebraically closedness like $\bar{\mathbf{Q}}$, that is, any non-trivial algebraic equation with coeffcients in complex numbers always has a solution in complex numbers (Gauss), and ii) closedness under taking limits where distance between two complex numbers z_1, z_2 is mesure by the absolute value $|z_1-z_2|$.

Furthermore, every proof of i) essentially uses a property, called the conformality of the product of complex numbers, where a germ of complex analytic functions can be found. Euler, who worked in 18th century, using complex numbers, showed already that the trigonometric functions and the exponential function, which were studied separately before, are combined by the beautifull relation $e^{iz} = \cos(z)+i\sin(z)$ (in particular $e^{\pi i} = -1$). Thus, the works of Gauss and Euler, titans in mathematics, established the role of complex numbers in mathematics. Then, there appeared several theories in physics, like elektro-magnetic theory, which are described by an essential use of complex number field. Even though what we observe are real numbers, quantum mechanics cannot be described without the use of complex number field. We have no choice of words but mysterious for the usefulness of complex numbers to describe laws of physics and the universe.

A complex number which does not belong to $\bar{\mathbf{Q}}$ is called a transcendental number. It was proved by Lindemann in 1882 that π is a transcendental number, using Euler's identity $e^{\pi i} = -1$ and the theory of approximations of the analytic function $\exp(z)= e^z$ by rational functions, which I will explain in §7. Returning to a question at the end of §3, we observe now that algebraic numbers $\bar{\mathbf{Q}}$ alone are not suffcient to measure the world. However, the complex number field **C**, likewise **R**, carries Cantor's problem stated in ∗4, and the question whether all complex numbers are necessary or only a very thin part of it is suffcient remains unanswered.

§7 Rational functions and analytic functions

So far I have described systems of numbers. It is not just to give an overview of the history, but because $\overline{\mathbf{Q}}$ and **R** themselves carry profound actual problems yet to be understood. Another reason is that the development of the concepts of numbers repeatedly became models of new mathematics. For example, let us consider the collection of polynomials in one variable *z*, denoted by **C**[*z*]. Similar to **Z**, it admits addition, subtraction and multiplication between its elements but not division. As we constructed rational numbers from integers, we consider a function which is expressed as a fraction $P(z)/Q(z)$ of two polynomials, called a *rational function*, and the whole of them, denoted by **C**(*z*). Then similarly to constructing a real number from **Q**, we consider a function which is a limit of a sequence of rational functions (in a suitable sense) and call it an *analytic function*. Let us denote the collection of such analytic functions by $\widehat{\mathbf{C}(z)}$, mimicking the notation in ∗3. I think the study of $\widehat{\mathbf{C}(z)}$ is easier than that of $\mathbf{R} = \widehat{\mathbf{Q}}$ and expect that the understanding of $\widehat{\mathbf{C}(z)}$ helps that of $\mathbf{R}=\widehat{\mathbf{Q}}$ as well as of **C**. The reason is that an element of **R** (**C**) is a limit of sequences of (Gaussian) rational numbers that provides little clue for capturing it, while for an element of $\widehat{\mathbf{C}(z)}$ we have a clue, the variable *z*. For instance, we have some freedom to substitute a favorable value in the variable *z* as needed. Therefore, we contrast $\overline{\mathbf{Q}}$ with $\widehat{\mathbf{C}(z)}$ instead of contrasting $\overline{\mathbf{Q}}$ with $\mathbf{R} = \widehat{\mathbf{Q}}$ as in §5.

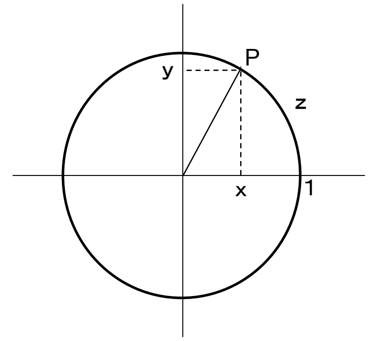


Figure 2: We consider a point P on the circle of radius 1 in the *x*-*y* plane. Then the length (angle) of the arc $\widehat{1\mathrm{P}}$ is given by the integral

$$z = \widehat{1\mathrm{P}} = \int_1^{\mathrm{P}} \sqrt{dx^2 + dy^2} = \int_1^x \frac{|\,dx\,|}{\sqrt{1 - x^2}}$$

Then as the inverse of the function $x \mapsto z$, we obtain the trigonometic-function $x = \cos(z)$.

§8 Transcendental functions and period integrals

An element of $\widehat{\mathbf{C}(z)}$ which is not either a rational function or an algebraic function (in a suitable sense) is called a *transcendental function*. The gamma function $\Gamma(z)$ and zeta function $\zeta(z)$ are examples of them. However, in what follows let us discuss about transcendental functions belonging to different category, namely their Fourier duals.

The exponential function exp(*z*) and the trigonometric functions, we have already seen, are, from a certain viewpoint, the first elementary transcendental functions appearing after rational functions. Let us briefly explain the reason. We learn in high school that the length of an arc of the unit circle can be obtained as the integral $z = \int_1^x \frac{|dx|}{\sqrt{1-x^2}}$ (Fig. 2). For the correspondence (or map) $x \rightarrow z$ defined by the integral, its inverse map $z \rightarrow x$ is the trigonometric function $x = \cos(z)$. In other words, the trigonometric functions are obtained as the inverse functions of the arc integrals over a circle (a quadratic curve). As we learn in high school, they are periodic functions with period 2π and satisfy the addition formulas (in particular, we can obtain the coordinates of the points that divide the arc equally into *q* parts for a natural number *q*, by solving an algebraic equation of degree $\leq q$). Then, arc integrals

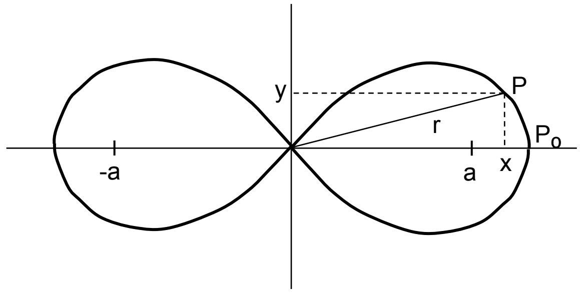


Figure 3: For a positive number a, the lemniscate curve is characterized as the loci of point P where the product of distances from two points $\pm a$ on the x-axis is the constant equal to a^2, and is given by the equation $(x^2 + y^2)^2 = 2a^2(x^2 - y^2)$. The length z of the arc $\overset{\frown}{P_0P}$ on the lemniscate is given by the integral

$$z = \overset{\frown}{P_0P} = \int_{P_0}^{P} \frac{dr}{\sqrt{1-r^4}}$$

where $r = \sqrt{x^2+y^2}$. Then, as the inverse of the function $x \mapsto z$, we obtain an elliptic function $r = \varphi(z)$ of period **Z**+**Z**i.

for curves of higher degrees and their inverse functions are natural subject of study. The theory of elliptic functions and abelian functions was born in that way.*6 The length of arcs in a lemniscate curve (see Fig. 3) is given by $\int \frac{dr}{\sqrt{1-r^4}}$. This was the first studied elliptic integral, 100 years before Gauss, when an Italian, Fargano, found a formula for the duplication of arc length of the lemniscate, and later Euler did the addition formulas (Jacobi approved it for the start of the theory of elliptic functions). The inverse function of the lemniscate integral is, from a modern viewpoint, an elliptic function having Gaussian integers **Z**+**Z**i as its periods.

§9 Kronecker's theorem = the first contact point between algebraic numbers and transcendental functions?

Nowadays the following two statements are known as Kronecker's theorems (we refer readers to the text book in ∗5 and ∗6 for terminology):

1. Any abelian extension field of the rational number field is obtained by adjoining values that are substitutions of rational numbers p/q to the variable z of the exponential function $\exp(2\pi iz)$ (for short, the coordinates of the points of the circle $S^1 = \{z \in \mathbf{C} \mid |z| = 1\}$ that divide it equally into q parts, see figure 1) to the field of rational numbers.
2. Any abelian extension of the gaussian integers **Z** + **Z**i is obtained by adjoining the coordinate values of the points of the lemniscate that divide it equally, where the values are expressed by special values of the elliptic functions associated with the lemniscate.

Kronecker's theorems (whose proofs he did not leave behind) involve both number theory and transcendental functions related to algebraic geometry. He devoted his later years of life to a proof of the advanced proposition that any abelian extension field of an imaginary quadratic field is obtained by adjoining solutions of the transformation equations for elliptic curves with complex multiplication. He called it "the dearest dream of my youth (mein liebster Jugend Traum)" in a letter to Dedekind, a German contemporary mathematician, when he was 58 years old.

It is said that Kronecker had many likes and dislikes; "God made the integers, all else is the work of man (Die ganzen Zahlen hat der liebe Gott gemacht, alles andere ist Menschenwerk)." is his saying. According to books of the history, he thoroughly attacked the set theory of Cantor, a contemporary German mathematician; Cantor was distressed with this and entered a mental hospital. Though Kronecker's mathematics that treats the exquisite structure of numbers, and Cantor's that was reached by thorough abstraction of those structures (see ∗4) are quite in contrast, I am attracted by both of their thoroughness, and the unhappy relation between them perplexes me. One may think Kronecker is on the side of $\overline{\mathbf{Q}}$, but I think this is a one sided opinion. His results or dreams turn out to tell about some delicate points where $\overline{\mathbf{Q}}$ and

$\widehat{\mathbf{C}(z)}$ contact. Kronecker's Jugendtraum was later solved by Takagi Teiji in Japan and others, with the building of class field theory.

§10 New dream

Following Kronecker, let me write about a dream of my own. Roughly speaking, Kronecker found that the first step (i.e. abelian extension of $\mathbf{Q}$) of extending the rational number field $\mathbf{Q}$ to its algebraic closure $\overline{\mathbf{Q}}$ corresponds to another first step (i.e. exponential function) of extending the field of rational functions $\mathbf{C}(z)$ to the field of analytic functions $\widehat{\mathbf{C}(z)}$ in such a manner that the algebraic extension is recovered by adjoining special values of the transcendental function. Let us expect, though we have no evidence so far, that similar correspondences between algebraic numbers and transcendental functions exist further, and that it causes certain "hierarchies" among the corresponding transcendental functions.[*7] Then the problem is what transcendental functions should appear.

My Jugendtraum is to construct (some candidates for) such transcendental functions by period integrals and their inverse functions, just like that the classical circle integrals and elliptic integrals gave birth to exponential and elliptic functions. To do it, I proposed the theory of primitive forms and their period integrals as a higher-dimensional generalization of the theory of elliptic integrals. More precisely, we have introduced a) semi-infinite Hodge theory (or non-commutative Hodge theory) in order to define the primitive form associated with a Landau-Ginzburg potential, b) torsion free integrable logarithmic free connections to describe the period map, c) the flat structure (or Frobenius structure) on the space of automorphic forms given as the components of the inverse maps of period integrals, d) several infinite-dimensional Lie algebras (such as elliptic Lie algebras, cuspidal Lie algebras, ...) in order to capture primitive forms in (infinite) integrable systems which are associated with a generalized root system and with a regular weight system, and e) derived categories for giving a categorical Ringel-Hall construction of those Lie algebras (every one of them is unfinished). It is mysterious that some pieces of these structures I have considered from purely mathematical motivations have come to be observed in topological string theory in recent physics. I sincerely wish these attempts for understanding of the system of numbers $\mathbf{C}$ should also lead to the understanding of the physics of the universe.

*1 To be precise, we don't allow division by 0.

*2 Let us explain a bit more precisely. The collection of rational numbers is equipped with an ordering. Then, for two numbers x and y, we define the distance between them by $|x-y| = \max\{x-y, y-x\}$ and regard them being closer to each other when the distance between them becomes smaller. We say that a sequence $y_1, y_2, y_3, \ldots$ approximates a number x if $|x-y_n|$ ($n = 1, 2, 3, \ldots$) becomes smaller and closer to 0. We say that an infinite sum (called a series) $y_1+y_2+y_3+\cdots$ converges to x and write $x = y_1+y_2+y_3+\cdots$, if the sequence $y_1, y_1+y_2, y_1+y_2+y_3, \ldots$ approximates x. E.g. $\pi^2/6 = 1+1/2^2+1/3^2+\cdots$ (Leibniz).

*3 One may denote $\mathbf{R}$ by $\widehat{\mathbf{Q}}$ in the sense that it is an analytic closure of $\mathbf{Q}$. However $\mathbf{Q}$ is also equipped with another distance than that in *2 called p-adic non-Archimedean distance for each prime number p, and we need to distinguish $\widehat{\mathbf{Q}}$ from the closure $\mathbf{Q}_p$ with respect to the p-adic distance.

*4 Cantor found that, forgetting the structures on the sets $\mathbf{N}$, $\mathbf{Z}$, $\mathbf{Q}$, $\overline{\mathbf{Q}}$, one can construct a one-to-one map between any two of them, while the set $\mathbf{R}$ is properly larger than them. This left the problem whether an intermediate size between $\mathbf{N}$ and $\mathbf{R}$ exists. Although Cantor himself proposed the continuum hypothesis that asserts no intermediate exist, now it is known that the continuum hypothesis is independent of the axioms of set theory. Namely we don't know whether there exists a subset of $\mathbf{R}$ which is properly smaller than $\mathbf{R}$ and properly larger than $\mathbf{N}$ or not.

*5 $\overline{\mathbf{Q}}$ is a union of subfields $\mathbf{Q}(\xi)$, called number fields obtained by adjoining finitely many algebraic numbers ξ to $\mathbf{Q}$. The projective limit: $\lim \mathrm{Gal}(\mathbf{Q}(\xi)/\mathbf{Q})$ of Galois groups corresponding to Galois fields $Q(\xi)$ (where ξ is closed under conjugation) is called the *absolute Galois group*. It is equipped with the inclusion relation among subgroups (hierarchy structure) corresponding to extensions of number fields. Reference: Emil Artin, *Algebra with Galois Theory*, American Mathematical Society, Courant Institute of Mathematical Sciences.

*6 We refer the reader to one best text on elliptic functions and period integrals from analytic viewpoint by C.L. Siegel: *Topics in complex function theory*, Part 1, Teubner (1970)).

*7 Hilbert has suggested certain automorphic forms as such transcendental functions for real quadratic fields. However, the author does not know whether it is reasonable to expect further such correspondeces. If there exist such correspondences, such transcendental functions form quite a thin ($\aleph_0$) subset of the field of all transcendental functions. Those functions should be, in spite of their transcendency, special functions which are controlled by an algorithm in a suitable sense. We can imagine many things, Moonshine for instance. What happens on the side of transcendental functions corresponding to algebraic extensions with non abelian simple groups as their Galois groups. But I don't think we have examples to assert mathematical propositions. Can we "resolve" Cantor's problem (see *4) considering only such a thin set of special transcendental functions and their special values? For the description of mathematics and physics of the universe, is such a thin set of transcendental functions sufficient?