FEATURE

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Quantum Field Theory and Mathematics

What are we made of? This is a long-standing question for human beings. Our body is made of cells, which are made of molecules. Molecules are made of atoms, which, in turn, are made of electrons and nuclei. The latter are made of neutrons and protons, which are finally made of quarks. Electrons and quarks are known as elementary particles, and cannot be further subdivided as far as we know at present. The forces between these particles include the electromagnetic force, about which you are probably familiar, and the *strong force*[*1] which binds quarks into protons and neutrons. Elementary particles and the forces among them are described by a framework called *quantum field theory* in theoretical physics.

As I will explain soon below, the computations based on quantum field theory reproduce many experimental results extremely well. At the same time, our understanding of quantum field theory is quite incomplete. Still, quantum field theory has been stimulating the development of various areas of mathematics. I would like to say something about this mysterious gap in our understanding of quantum field theories.

Quantum field theory is incomplete

What do I mean when I say quantum field theory is incomplete? Let us compare the situation of quantum field theory with that of general relativity and quantum mechanics, both of which appeared in the early twentieth century. You might know them as difficult subjects, but in fact they are rather well understood as theoretical frameworks, by physicists and mathematicians. First, there are many textbooks aimed at physics students, which can be read alone in principle. Second, it is possible to express these frameworks to mathematicians in single sentences: we can just say: “general relativity is about studying the Einstein equation on Riemannian manifolds” and “quantum mechanics is the study of self-adjoint operators on Hilbert spaces.” The point here is not about whether or not you can understand these two sentences, but the fact that there is a way to tell mathematicians what they are in a concise way.

Now, what is the situation with quantum field theory? There are many textbooks for physics students, but they are rather difficult to study alone. Furthermore, there is no way to tell mathematicians in a few sentences what quantum field theory is. There might be no need for every physics theory to be understood by mathematicians, but the fact that physicists cannot communicate it in a straightforward way to mathematicians should suggest that physicists themselves do not understand it well enough.

But then, what do I think quantum field theory is?[*2] For me, the framework called quantum field theory is merely a random collection of calculational techniques and results which I learned through textbooks and various original papers. I do not have a logical, uniform and straightforward understanding of it. And it seems it is not just a problem unique to me. For example, when you open a textbook on quantum field theory and start studying it, you often encounter strange statements such as: "the explanation given in the last chapter to the concept X was not quite right. In fact the correct statement is the following." You continue reading and then find: "in the last chapter we said that the correct interpretation of the concept X was such and such. But that is not perfectly true either. In reality it is..." These things rarely happen in the textbooks on general relativity and quantum mechanics.

Quantum field theory works very well

Still, computations done using quantum field theory reproduce experimental results quite well. For example, the anomalous magnetic moment of an electron, which is the strength of an electron as a magnet, can be computed in terms of the expansion in the fine structure constant, which is the basic strength of the electromagnetic force. Its theoretically computed value and the experimentally observed value are in extremely good agreement.[*3]

Let us next discuss the case of the *strong force*. For theoretical physicists, this is defined in terms of the so-called path integral, which is an infinite-dimensional integral. Various physical quantities can be computed by performing this process. However, it is impossible to perform the integration an infinite number of times in practice. Instead, the result is computed by first making an approximation by a finite sum and then taking the limit numerically. This procedure is now carried out using the world's fastest supercomputers. In the last ten years, the output of the computation started to show good agreement with experimental results.[*4]

One of the Clay Millennium Problems[*5] is essentially equivalent to proving that this limiting procedure converges. We now know that the numerical value, before actually taking the limit, already agrees quite well with reality.

There are many other cases where the computations done using quantum field theory agree well with experimental results, although there is no satisfactory formulations of quantum field theory. It is expected, therefore, that some well-defined mathematics would be extracted from quantum field theory, just as Euclidean geometry was abstracted from various technical progresses in ancient Egypt and Babylonia.

Various exiting formulations of quantum field theory

Of course, there have been many researchers who thought exactly in the same way. After all, quantum field theory itself has been studied for about one hundred years already. A very early effort in the 1950s is now known as *axiomatic quantum field theory*, which successfully axiomatized the aspects of quantum field theory then known. Unfortunately, not much of the later developments on the physics side have been incorporated.[*6] Then in the 1980s, a couple of formulations were given to subclasses of quantum field theory which are

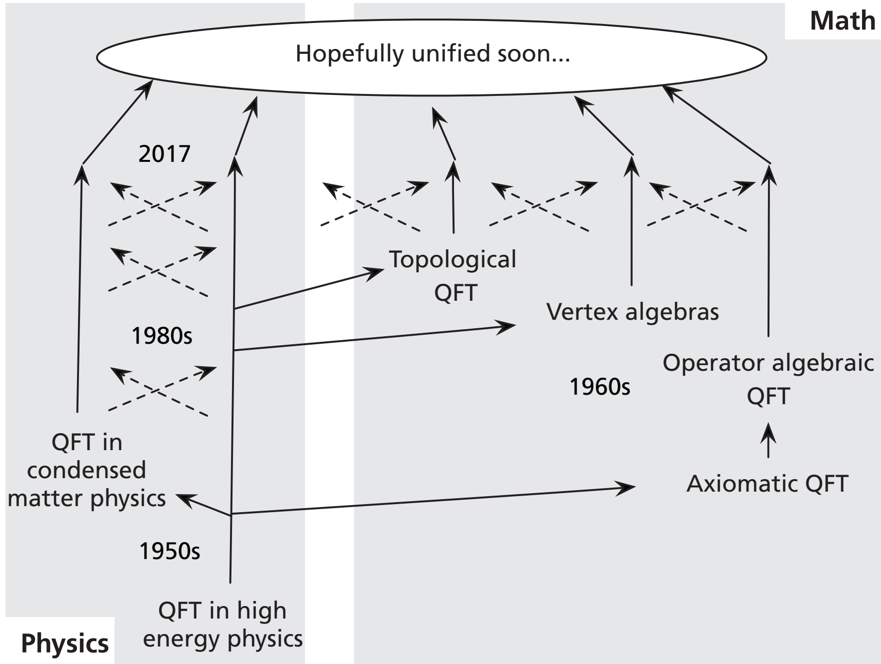


Figure 1: Existing formulations of quantum field theory

amenable to rigorous mathematical treatment, such as *topological quantum field theory* and *vertex algebras*. These cannot, however, deal with the quantum field theory which describes our microscopic world. Also, these formulations, once mathematical definitions are given, became separate subdisciplines of mathematics and had their own developments, with not much communications between them. Happily, we are starting to see fruitful interactions among them in the last ten years. I summarized the interrelationship of these formulations in Figure 1.

In addition, the quantum field theory which actually describes the real-world elementary particles is usually described in physics textbooks in terms of the path integral, which is infinite-dimensional. Therefore, a long-standing idea on the proper mathematical formulation of quantum field theory is that we need to justify and make rigorous this integral. This approach is known as *constructive quantum field theory*. However, in the last ten years, it is recognized on the physics side that there are many examples of quantum field theories[*7] which do not seem to be described by the path integral. This means that the completion of the program of the constructive quantum field theory does not mean a successful mathematical formulation of quantum field theory.

Mathematical applications of quantum field theory

So far I emphasized that we do not know how to formulate quantum field theory mathematically. Still, there have already been many rigorous mathematical results inspired by the research in quantum field theory. For example, from the study

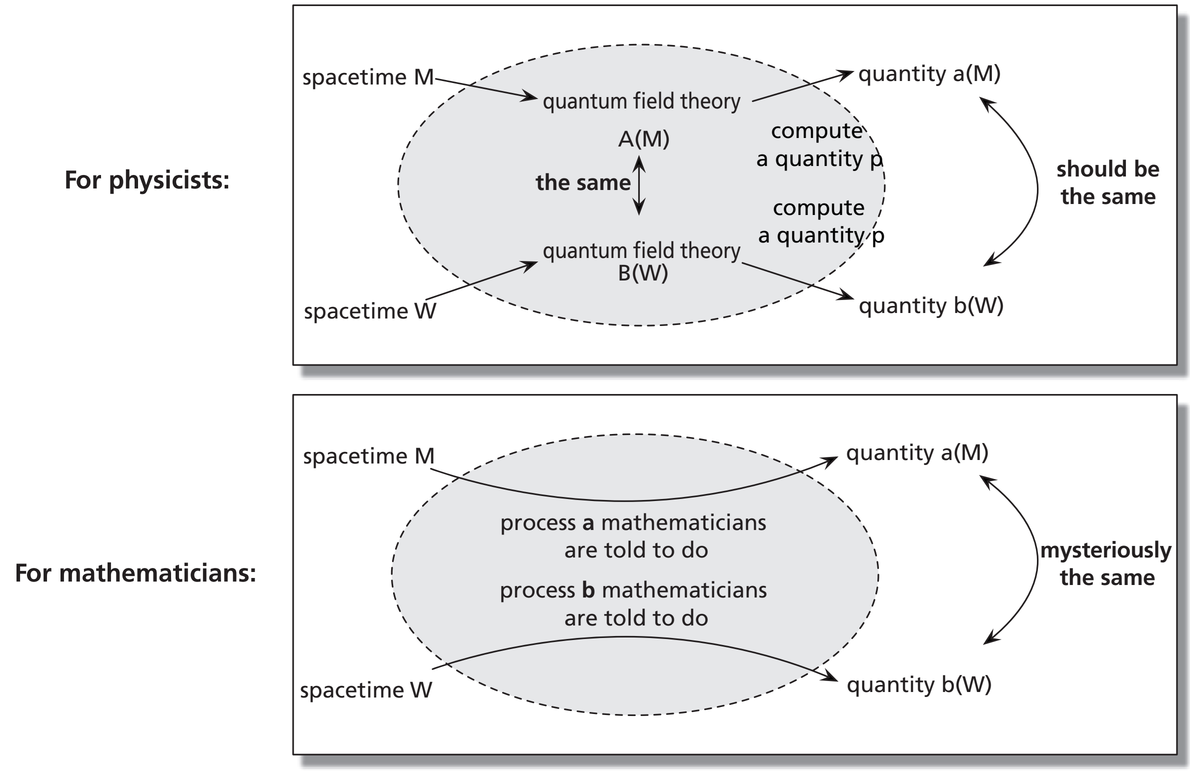


Figure 2: Mathematicians do not understand the area within the dotted line.

of two-dimensional quantum field theory in the early 1990s, there arose a subarea of mathematics known as *mirror symmetry*. Also, stimulated by the physics results of Seiberg and Witten concerning four-dimensional quantum field theory, our understanding of the topology of four-dimensional manifold was greatly improved around 1995. This is called the *Seiberg-Witten theory* in the mathematical literature.[*8] In my own collaboration in theoretical physics with Luis F. Alday and Davide Gaiotto around 2010, we find that there should be a relation between the geometry of the instanton moduli space and the representation theory of infinite dimensional algebras. This conjecture was soon mathematically formulated, which got other mathematicians interested and inspired them to rigorously prove it.

In a sense, these can all be considered as an application to mathematics of quantum field theory. However, these mathematical works are usually done quite independently from the mathematical sub-disciplines which deal with formulations of quantum field theory. Why is there such a mismatch? The reason can be understood by looking more closely at how these applications arose. Let us take mirror symmetry as an example.

How mathematical applications are extracted

In superstring theory, there are two types of strings, called type IIA and type IIB. The motion of these strings within a spacetime M is described by a quantum field theory depending on the type, $A(M)$ and $B(M)$. Slightly later, it was realized that there is a duality where a type IIA string moving in a spacetime M is equivalent to a type IIB string

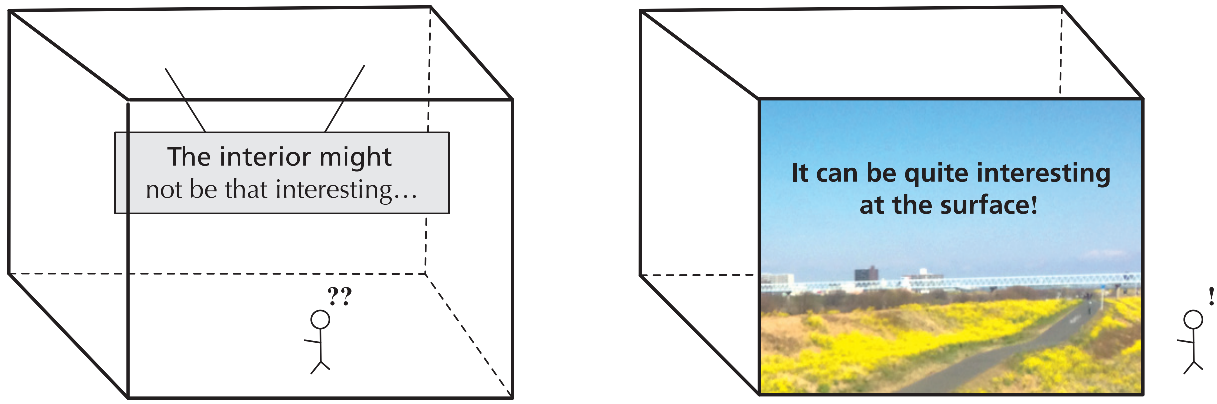


Figure 3: Interesting things can happen at the surface even when the interior is not that interesting.

moving in a different spacetime W. Then there should also be an equality $A(M) = B(W)$ between the quantum field theories describing them. Now, various quantities can be computed from a quantum field theory. Let us take the probability p of an event. Then of course we have $p(A(M)) = p(B(W))$.

Here, $A(M)$ and $B(W)$ are (un)fortunately not the kind of quantum field theories already formulated in mathematics. Therefore mathematicians do not understand them. However, it is still possible for them to understand the computational process of obtaining $a(M) = p(A(M))$ from M without understanding quantum field theory. Similarly, they can understand the process of computing $b(W) = p(B(W))$. Still, the processes a and b look totally unrelated if the intermediate steps, which used quantum field theory, are hidden from view. This makes these correspondences very mysterious from the mathematician's point of view. Please see Figure 2 for an illustration.

The important step is to translate what existing mathematics can deal with into quantum field theory which are still ill-defined, and then to translate it back to objects which can be analyzed again using existing mathematics. Many other mathematical applications of quantum field theory arose basically in the same manner, not just mirror symmetry. Mathematicians feel that a mysterious new result is obtained, because what is equivalent from the standpoint of quantum field theory looks totally different from the viewpoint of the mathematics currently available. The existing mathematical formulations of quantum field theory I explained above are not powerful enough to be used to study these cases.

What should we do about formulations of quantum field theory?

How can this situation be improved? Can there be a way to convey the content within the dotted line in Figure 2 to mathematicians? For this, we need to clarify what quantum field theory is and what properties they satisfy. If this can be done, this will be good not only for mathematicians but also for physicists. This is because these general properties of quantum field theory are not yet written down in textbooks in any concise manner, even within theoretical physics.

It is true that these properties can be found scattered in various academic papers. The most crucial parts, however, can only be found in the minds of the physicists who are actually carrying out the research, and they are shared only vaguely among them. This is clearly an unsatisfactory situation. What should we do if a big earthquake or a terrorist attack hits a major international conference? How do we reconstruct such knowledge if it is not even written down? This thought alarmed me, and so I started to write these things down a few years ago. But soon, after I wrote about a hundred pages, I found that it was not yet ripe for me to start this task.

I realized this problem while I was learning recent developments in condensed matter physics, where it was shown that there are quantum field theories which show extremely rich properties even though they look almost completely empty to the untrained eyes of high energy physicists, including mine. In condensed matter physics, we need to consider experimental samples, which necessarily have surfaces and boundaries. There might be nothing particularly interesting within the sample, but there can be rich phenomena at its boundaries and surfaces. Correspondingly, there can be quantum field theories which are almost empty in the bulk, and still have rich physics at the boundaries and surfaces. Please see Figure 3.

This is a natural idea in condensed matter physics, but it might have been a blindspot for high energy physicists. At least that was the case with me. This might be due to the historical background: the original aim in high energy physics was to identify the quantum field theory which describes the real world at the microscopic level. Then, that quantum field theory exists everywhere in this world, or more simply, that specific quantum field theory is the world, so there are no boundaries to it.

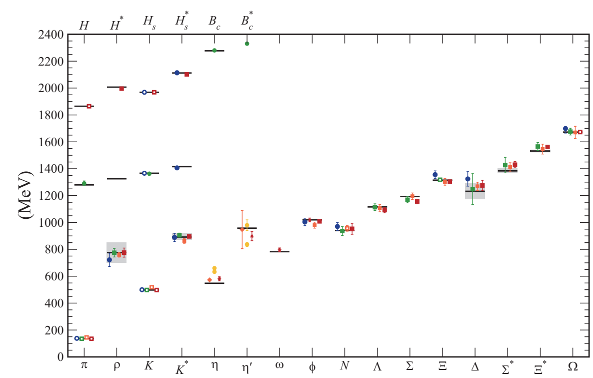
I learned of these new points of view only in the last few years, and this was a big surprise to me: something I thought would be almost trivial turned out to be not trivial at all. I am planning to spend the next few years at least to digesting and internalizing these developments. Once this is done, I should be able to restart writing down in one place my own understanding of quantum field theory. But when will that happen? Only time will tell.

*1 The phrase *strong force* should be thought of as a proper noun that names a specific type of a force, and not just a generic phrase meaning a force which happens to be strong.

*2 I would be usually counted as a physicist, so I guess I have to express my opinion as a physicist, rather than as a mathematician. But I should mention here that real physicists do not consider me as a physicist because what I study has no direct relationship with the real world, and that real mathematicians do not consider me as a mathematician because I do not rigorously formulate questions and prove theorems.

*3 See, e.g., Tatsumi Aoyama, Masashi Hayakawa, Toichiro Kinoshita, Makiko Nio, Tenth-Order QED Contribution to the Electron g-2 and an Improved Value of the Fine Structure Constant, arXiv:1205.5368 [hep-ph]. Both experiments and computations give the value $a_e \simeq$ 0:001159652180.

*4 See, e.g., Andreas S. Kronfeld, Twenty-First Century Lattice Gauge Theory: Results from the QCD Lagrangian, arXiv:1203.1204 [hep-lat], from which the following figure is taken. The horizontal bars are the measured values of the mass of the mesons, and dots in various colors are the computational results of various research groups.



*5 Posed in 2000 by the Clay Mathematical Institute. Each comes with a $1,000,000 prize. See http://www.claymath.org/millennium-problems

*6 One small example is the following. In quantum field theory, there are concepts called *gauge symmetry* and *flavor symmetry* now on the physics side. Around the 1950s, the same concepts were known as local gauge symmetry and global gauge symmetry, and this terminology is still used in the community of axiomatic quantum field theory. It is a trivial issue, but with many trivial differences, it becomes hard to communicate across sub-disciplines.

*7 The phrase 'quantum field theory' can either mean the entire framework or individual examples within that framework. Here it is used in the latter sense.

*8 The phrase 'the Seiberg-Witten theory' means the physics results for physicists and the mathematics results for mathematicians, and they are quite distinct. If you want to learn about one side but borrow a book about the other side, you will be totally at a loss. This happened to me more than once.