What Is a WDVV-Algebra?

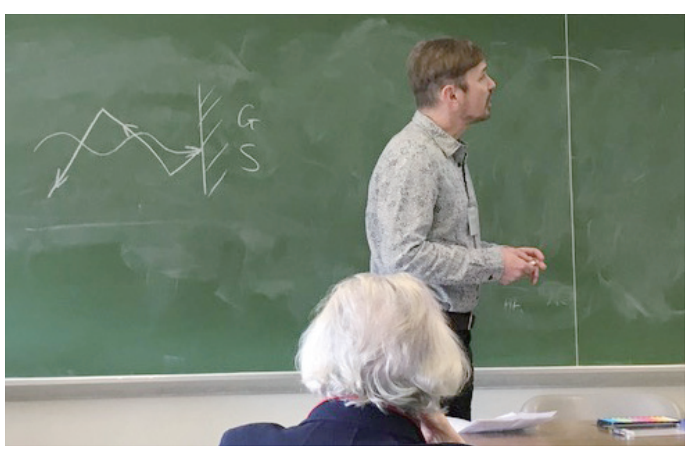
Alexander A. Voronov

Professor, School of Mathematics, University of Minnesota
and Kavli IPMU Visiting Senior Scientist

People often ask me what a Witten-Dijkgraaf-Verlinde-Verlinde (WDVV) algebra is. And I never hesitate to tell them. After all, the definition is as beautiful, as it is useful. And I have embedded a mnemonic rule in it, so that if you see this definition once, you will never forget it! Here we go. A *WDVV-algebra* is a graded vector space V with graded symmetric multilinear operations $(v_1, v_2, \dots, v_n)$ of degree $2(n-2)$, one for each $n \geq 2$, such that they satisfy the following associativity condition. Define a formal deformation of the commutative product (v_1, v_2) on V :

$$(w, d)_v := \sum_{k=0}^{\infty} \frac{1}{k!} (w, d, v, v, \dots, v)\, \lambda^k, \quad v \text{ appearing } k \text{ times in the } k \text{ th term,}$$

for every "Witten" $w \in V$, "Dijkgraaf" $d \in V$, and all the "Verlinde brothers" $v \in V$, where λ is a formal parameter.[1] Then the condition is that the deformed bilinear product $(w, d)_v$ must be associative. Apparently, because of some controversy associated with the name, this structure is more often called a *hypercommutative algebra*.[2] It is also equivalent to the structure of a *linear Frobenius manifold* on V.



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[1] See, λ-Adic Topology on p. 22 in this issue.

[2] See Ezra Getzler, **Operads and moduli spaces of genus 0 Riemann surfaces**. *The moduli space of curves (Texel Island, 1994)*, 199–230, Progr. Math., 129, *Birkhäuser Boston, Boston, MA*, 1995.