What Is a WDVV-Algebra?

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People often ask me what a Witten-Dijkgraaf-Verlinde-Verlinde (WDVV) algebra is. And I never hesitate to tell them. After all, the definition is as beautiful, as it is useful. And I have embedded a mnemonic rule in it, so that if you see this definition once, you will never forget it! Here we go. A WDVV-algebra is a graded vector space $V$ with graded symmetric multilinear operations $(v_1, v_2, \ldots, v_n)$ of degree $2(n – 2)$, one for each $n \geq 2$, such that they satisfy the following associativity condition. Define a formal deformation of the commutative product $(v_1, v_2)$ on $V$:

$$(w, d)_k := \sum_{k=0}^{\infty} \frac{1}{k!} (w, d, v, \ldots, v) \lambda^k, \text{ } v \text{ appearing } k \text{ times in the } k \text{ th term},$$

for every “Witten” $w \in V$, “Dijkgraaf” $d \in V$, and all the “Verlinde brothers” $v \in V$, where $\lambda$ is a formal parameter.\(^1\) Then the condition is that the deformed bilinear product $(w, d)_k$ must be associative. Apparently, because of some controversy associated with the name, this structure is more often called a hypercommutative algebra.\(^2\) It is also equivalent to the structure of a linear Frobenius manifold on $V$.

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\(^1\) See $\lambda$-Adic Topology on p. 22 in this issue.