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Research Area: Mathematics

Analogy and Mathematics

In March, 1940, in the midst of war's chaos, a mathematician, arrested for refusing military service, wrote a 14-page letter to his sister, a philosopher, from Bonne-Nouvelle prison at Rouen in France. He wrote, "On the one hand the analogy (of function fields over finite fields) with number fields is so strict and obvious... while, on the other hand, the one between the function fields (over finite fields) and the "Riemannian" fields... is to profit in the study of the first from knowledge acquired about the second, and of the extremely powerful means offered to us..."[*1] The mathematician is André Weil, who later proposed a surprising conjecture on zeta functions for varieties over finite fields following an analogue with the Riemann hypothesis,[*2] and has made a strong impact on mathematics up until now.

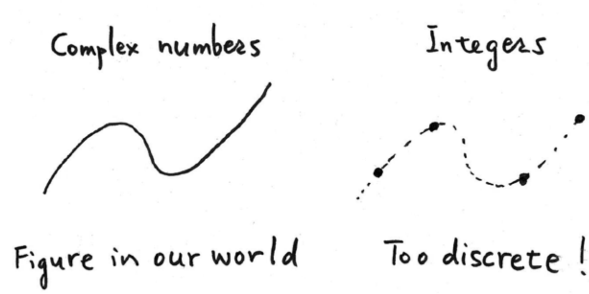

Figure 1. Realization of an algebraic variety in two different worlds.

§1 Weil's Philosophy

1, 2, 3,... Integers are one of the most fundamental and classical mathematical concepts that people are familiar with. It is an extremely difficult object to study, and modern mathematics is not powerful enough to answer many simple questions it poses. However, when we find a piece of truth in number theory, people tend to have a superb outcome.

From ancient times, solving equations has attracted the attention of lots of mathematicians. Solution to quadratic equations $x^2 + ax + b = 0$ was known already by ancient Babylonians, presumably discovered by practical needs. Attention changed to more complicated equations as time went by. When we study equations, one way is to think of them graphically. For example, consider the equation $y = x^2$. We learn that this equation represents a parabola, which enables us to study the equation geometrically. Generalizing this approach, algebraic geometry is a branch of mathematics that tries to consider systems of multi-variable equations geometrically. Algebraic geometry has already been discussed several times in the *IPMU News* (e.g. Toda[*3] and Bondal[*4]). Figures defined by systems of equations are called algebraic varieties in algebraic geometry. Algebraic geometry is situated at the intersection of various fields of mathematics. Given

[*1] Refer to Column 1 for some explanation of fields which appear in this sentence and rings which appear in §2. Finite fields are discussed in detail in §2. Riemannian fields are fields related to complex geometry, and often called function field over the field of complex numbers.
[*2] For the Riemann hypothesis, see §3.
[*3] Yukinobu Toda, *Kavli IPMU News* No. 20 (2012) p. 4.
[*4] Alexey Bondal, *IPMU News* No. 14 (2011) p. 4.

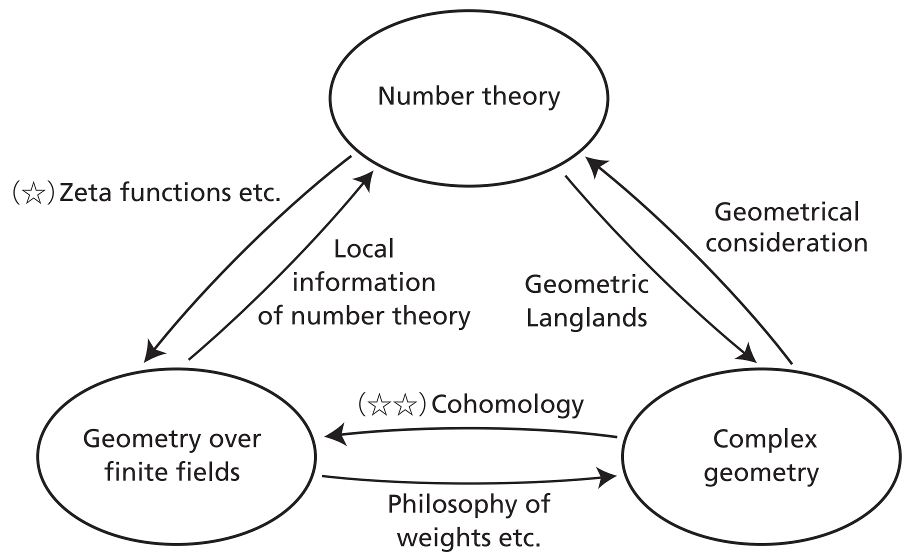

Figure 2. Weil's Trinity.

algebraic varieties, we may consider figures defined by integral solutions. Studying such solutions could be seen as a part of number theory. We may also consider figures defined by complex solutions of algebraic varieties. Now, this is a realm of complex geometry.

When we consider solutions in different places, their landscapes are totally different. For example, in complex geometry, algebraic varieties can be considered as figures in higher-dimensional complex space, so that geometric thinking is possible. But, if we wish to study integral solutions, the figure defined by such solution is too discrete to use geometric intuition (see Figure 1).

These are different worlds defined by the same language, algebraic geometry. Surprisingly, Weil's philosophy (which he himself admits was not the first to assert) tells us that these seemingly unrelated worlds have relations beyond our perception, called *analogy*, and we can attain the truth of mathematics when we think of them as the Trinity shown in Figure 2.

§2 Zeta Functions

The Riemann zeta function is the function defined by a simple series

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

In the mid-19th century Riemann found out that information on the distribution of prime numbers is encrypted in this function, and he proposed an ultimate form of this expected information. He formulated it into a celebrated conjecture, which we now call the *Riemann hypothesis*. This conjecture is still a central problem in mathematics, but it seems that we are far from its resolution.

To compare with integers, polynomial rings over finite fields have often been considered. To introduce these objects, let us define finite fields first. Let p be a prime number. Consider the set $\{\overline{0}, \overline{1}, \ldots, \overline{p\text{-}1}\}$ denoted by $\mathbf{F}_p$. In fact, we may define four elementary operations (addition, subtraction, multiplication, and division) for the elements of this

Feature

set just as we define them for the elements of the set of rational numbers (a rational number is a real number that can be written as a simple fraction). This is nothing difficult, and for addition and multiplication, all we need to do is to consider the remainder after division by p. For example, when $p = 7$, we have

$$\overline{4} \times \overline{5} = \overline{20} = \overline{6}.$$

Since $\overline{20}$ is not defined, the meaning of the equation is a bit ambiguous, but we leave the interpretation of it to the readers. Division is slightly more problematic, but when we want to compute

$$\overline{4} \div \overline{5} = \mathbf{?},$$

all we need to do is to find $\mathbf{?}$ so that

$$\overline{5} \times \mathbf{?} = \overline{4}.$$

The interested reader can check that the answer is $\overline{5}$. If we consider p which is not a prime number, we can define multiplication and addition in the same way, but we cannot define division. For example, for $p = 12$ (the world of a clock!) it is easily understood that $\mathbf{?}$ does not exist such that

$$\overline{4} \times \mathbf{?} = \overline{1}.$$

The polynomial ring $\mathbf{F}_p[x]$ in question is the set of polynomials whose coefficients are elements of $\mathbf{F}_p$. For example, in $\mathbf{F}_3[x]$, we have the following 9 polynomials with degree less than or equal to 1:

$$\bar{0}, \bar{1}, \bar{2}, x, x+\bar{1}, x+\bar{2}, \bar{2}x, \bar{2}x+\bar{1}, \bar{2}x+\bar{2}.$$

Likewise, we have polynomials with degree bigger than 1, so that $\mathbf{F}_3[x]$ contains infinitely many polynomials. Similar to the additions and multiplications of the usual polynomials, additions and multiplications are defined for elements of $\mathbf{F}_p[x]$. Moreover, we have the notion of irreducible polynomials, which are polynomials which cannot be divided by polynomial with lower degrees. Irreducible polynomials in $\mathbf{F}_p[x]$ can be regarded as the notion corresponding to prime numbers in the set of integers.

Now, if the Riemann zeta function for the integer ring is too challenging to deal with, we may try to consider an analogous zeta function for its cousin ring $\mathbf{F}_p[x]$ and study it. To define the analogous function, an important discovery is the Euler's product representation

$$\zeta(s) = \prod_{p:\text{prime number}} \frac{1}{1-p^{-s}}.$$

This representation enables us to interpret the zeta function "algebro-geometrically." Let f be an element of $\mathbf{F}_p[x]$. We denote the degree of f by $\deg(f)$. By analogy, then, we may define

$$\zeta_{\mathbf{F}_p[x]}(s) = \prod_{f:\text{irreducible}} \frac{1}{1-p^{-\deg(f)\cdot s}}.$$

Here, the factor p^{-s} in the definition of the Riemann zeta function is replaced by $p^{-\deg(f)\cdot s}$. This is because, when we interpret p in the definition of $\zeta(s)$ as the "size" of the prime number p, it is reasonable to measure the size of f as $p^{\deg(f)}$. Algebro-geometrically, $\mathbf{F}_p[x]$ is understood to be a line (over a finite field). With this interpretation, all the factors appearing in the definition of $\zeta_{\mathbf{F}_p[x]}(s)$ have algebro-geometric meanings. Pursuing this, it is not hard to define the

Column 1: Fields and rings

A ring is a set such that addition and multiplication rules are defined. For example, the set of integers (⋯, -2, -1, 0, 1, ⋯) is closed under taking addition and multiplication in the usual sense, so we may say that the set forms a ring. The set of polynomials has similar property, so it is also an example of a ring. Fields are a special kind of ring. More precisely, a ring is said to be a field if any element but 0 has an invertible element. Finite fields are examples of fields as well as the set of rational numbers and the set of real numbers.

zeta function $\zeta_X(s)$ for algebraic variety X over a finite field. This is the zeta function defined by Weil, and nothing but (☆) of the trinity shown in Figure 2.

§3 Weil Conjecture

Weil computed $\zeta_{\mathbf{F}_p[x]}(s)$, or more generally $\zeta_C(s)$ for general curves C. It showed that, contrary to Riemann's original zeta function, the zeta functions $\zeta_{\mathbf{F}_p[x]}(s)$ and $\zeta_C(s)$ are rational functions (namely fractions of polynomials). More precisely, we may write

$$\zeta_C(s) = \frac{f_1(p^{-s})}{f_0(p^{-s}) \cdot f_2(p^{-s})}$$

where f_n are polynomials. Even more surprisingly, the roots of the equation $f_n(t) = 0$ have their absolute value equal to $p^{n/2}$. The absolute value could be anything, but transcendental power forces us to pick extremely specific rational numbers $n/2$! Weil thought he knew what these numbers were. The Riemann hypothesis conjectures that the Riemann zeta function has non-trivial zeros only on the line Re(s) = 1/2, and the half integers could be seen as a perfect analogue of this. The original Riemann conjecture is too hard to tackle, but an analogue for curves over finite fields was formulated and proven by Weil.

Moreover, he found out that $f_n(t)$ has geometric information. Namely, for him, computing $f_n(t)$ seemed as if to compute the cohomology $\mathrm{H}^n(X)$.

Cohomologies express "topological information" of figures. In research, it is extremely important to describe characteristics of things. How do we extract characteristics of figures such as the one shown below?

Mathematical objects that characterize figures are called "invariants." For example, volume and surface area are typical invariants. These invariants readily change if we deform the figure. Of course, these invariants are important, but sometimes we need to express the characteristics of figures more roughly. In that case, we sometimes count the number of holes in the figure. This number doesn't change even if we stretch the figure. On the other hand, no matter how we stretch a sphere, we can't transform it into a torus. This implies that an invariant like the number of holes may be used to classify figures. Geometry, which aims to extract figures' characteristics which are invariant under continuous deformation (e.g. number of holes), is called "topology." and

Column 2: Invariants

Invariants means quantities invariant under certain operations. Why can we consider surface area as an invariant? There is a philosophy that geometry can be classified by invariance under transformations. This was first claimed by Klein in his famous Erlangen program, which is so to say a guideline of geometry. For example, Euclidean geometry is a geometry which is invariant under parallel translation or rotation, and topology is a geometry which is invariant under continuous deformation, which has more freedom than Euclidean geometry. Surface area is an invariant in Euclidean geometry, and we have so-called cohomologies as topological invariants. Of course, cohomologies can be seen as invariants for Euclidean geometry as well. The word "invariants" makes sense once we specify geometry.

cohomology is an invariant generalizing and abstracting the hole number. Cohomology is very important and frequently appears in geometry which deals with continuous objects, such as complex geometry. Geometry over a finite field stands on the other extreme. As finite fields have only finitely many elements, it doesn't make sense, a priori, to take continuous invariants. Nevertheless, Weil claimed that topological information is hidden in the zeta function. This is (☆☆) of the trinity (see, Fig. 2).

After these observations, Weil proposed conjectures on properties of the the zeta functions for more general varieties. Concretely, for any algebraic variety, $\zeta_X(s)$ is a rational function, and the absolute value of the root of each polynomial is equal to $p^{n/2}$ for some integer n. In addition to these conjectures, Weil predicted the existence of cohomology theories, which have the ability to extract topological information, for varieties over finite fields: he predicted it in 1949, shortly after the war.

Why are the zeta functions important? This is rather a philosophical question. People studying number theory "believe" that the zeta functions contain important information. However, this belief is not just fantasy. For example, the BSD (Birch and Swinnerton-Dyer) conjecture, which, as well as the Riemann hypothesis, has been offered a prize money of $1 million*5 for the solution to it, predicts that the zeta functions have information on the number of solutions of the defining equation of certain algebraic varieties. The numbers of solutions of systems of equations are the ultimate information that number theorists look for. Other than this, many difficult and central questions are related to the zeta functions, and we still believe that we may reach the truth by studying the zeta functions.

§4 Grothendieck and ℓ-adic Cohomology

In the late 50s, a genius, Alexander Grothendieck appeared to solve the Weil conjecture. He started to construct the cohomology theory that Weil had predicted. With the aid of M. Artin and others, after 10 years of concentration he succeeded in constructing topological cohomology theory for varieties over finite fields. It was called the ℓ-adic cohomology. The theory was far more general and abstract than Weil had imagined. Their results were collected in the seminar notes called SGA,*6 with the total number of pages more than 5000. He named the new geometry "arithmetic geometry."

Even though most of the Weil conjecture had been solved due to their efforts, the analogy with the Riemann hypothesis seemed to remain unsolved. However, Grothendieck's student, another genius, Pierre Deligne successfully solved the last piece of the Weil conjecture fully using the framework that Grothendieck had established. More than 30 years had passed since the letter of Weil in 1940.

What is intuitive in complex geometry could frequently be extremely hard in arithmetic geometry when following analogies with complex geometry. For this purpose, the utmost understanding of concepts of complex geometry was needed. Due to this fact, an enormous amount of mathematical notions and philosophy was yielded while arithmetic geometry was being constructed, and the influence of the geometry over a finite field to complex geometry cannot be overestimated. Hodge theory could be seen as an analogue of complex geometry (cf. Deligne's Hodge I*7), and the theory of weights coming from Hodge theory is indispensable in geometric representation theory. The geometric Langlands program is a theory considered by following an analogy between number theory and complex geometry via arithmetic geometry, and some experts point out relations with physics.

*5 In 2000, the Clay Mathematics Institute in the U.S. offered a $1 million prize for solving each of the seven mathematical problems including the Riemann hypothesis. Up to the present time, only the 'Poincaré conjecture' has been solved by Grigori Perelman.

*6 *Séminaire de Géométrie Algébrique*, http://library.msri.org/books/sga/.

*7 P. Deligne, "Théorie de Hodge. I," *Actes du Congrès International des Mathématiciens* 1. (1970) 425.

The notion of derived categories, which is being extensively investigated in algebraic geometry and which some of the Kavli IPMU researchers are interested in, is one of the numerous notions that were yielded in the process mentioned above. The return has turned out to be huge.

§5 ℓ-adic, p-adic, and the Future

Once again, let us come back to the analogy between complex geometry and geometry over finite fields. In complex geometry we have topological cohomology theory, but we also have analytic cohomology theory. These two cohomology theories have been known to coincide. Since Grothendieck constructed the ℓ-adic cohomology theory as an analogy with topological cohomology theory, it is also natural to expect an analogy over a finite field with analytic cohomology. In fact, Dwork had already considered such cohomology theory prior to ℓ-adic theory, and he had shown the rationality of the zeta function in the Weil conjecture. However, construction of a general theory like ℓ-adic theory was hard, and the theory fell behind ℓ-adic theory which was theoretically complete. There are several theories which should be mentioned such as Grothendieck's crystalline cohomology and Monsky-Washnitzer's cohomology, but finally, in the 80s, Pierre Berthelot constructed analytic cohomology theory for varieties over finite fields, called the rigid cohomology. This cohomology theory is sometimes called p-adic cohomology theory. Even though it was defined, many expected fundamental properties were left as conjectures. However, recent development of p-adic differential equation theory as well as the discovery of the weak desingularization theorem by de Jong finally allowed Kiran Kedlaya to establish p-adic cohomology theory. Using these results, I proved a Langlands-type theorem, which shows that, at least in the curve case, p-adic and ℓ-adic cohomology have essentially the same information. It realizes the philosophy of Grothendieck, which states: "All cohomologies stem from motives."

Column 3: ℓ-adic and p-adic theories

In arithmetic, when a prime number p is fixed we often distinguish other prime numbers different from p, and they are often denoted by ℓ. For example, consider a quadratic equation $ax^2+bx+c=0$. It is well-known that the roots are

$$\frac{-b \pm \sqrt{b^2-4ac}}{2a}.$$

Note that we have $2a$ as the denominator. Consider the equation in $\mathbf{F}_2$. Since $2=0$ here, the denominator doesn't make sense. On the other hand, it makes sense if we consider the equation in $\mathbf{F}_\ell\,(\ell \neq 2)$.

This shows that we have different behaviors as we vary prime numbers. Similar things often happen, which is the reason to distinguish prime numbers, and ℓ-adic and p-adic cohomologies are defined by very different methods.

I mentioned analysis over finite fields, but it is mysterious that imitation of analytic theory works over such discrete fields. I always have the impression that there is no reason the theory over finite fields should behave as if it were the real world. It seems as if some invisible power gave rise to the analogy. However, I don't want to cease the exploration just by worshiping the mysterious power, but want to make it a part of human knowledge by understanding it via the same language: analytic theory in complex geometry. When a new analogy between analysis for complex varieties and arithmetic geometry is realized, it should lead us to a deeper understanding of mathematics as an incarnation of the classical trinity.