Vainshtein mechanism in Horndeski’s general scalar-tensor theory (and in massive gravity)

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Based on work with
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Talk plan

✓ Introduction & Motivation
✓ Horndeski’s most general scalar-tensor theory
✓ Static, spherically symmetric, weak gravitational field
✓ Application
✓ Summary
Introduction

**Mystery of dark energy**

**Modified gravity** as an alternative to dark energy?

Modification would persist down to small length scales...

Need **screening mechanism** in the vicinity of matter

**Basic idea**

Extra d.o.f is **effectively weakly coupled to matter**

—— **Vainshtein mechanism**  
Vainshtein (1972)
Example

Cubic Galileon non-minimally coupled to matter:

\[
\mathcal{L} = \frac{1}{8\pi G} \left[ -\frac{1}{2} (\partial \varphi)^2 - \frac{r_c^2}{3} (\partial \varphi)^2 \Box \varphi \right] + \varphi T^\mu_\mu
\]

(\varphi : \text{dimensionless})

Key non-linearity

\[ r_c^2 \Box \varphi \text{ can be large even if } \varphi \ll 1 \]

\[
\varphi \sim \frac{r_g}{r} \ll 1, \quad r_c^2 \Box \varphi \sim \frac{r_c^2 r_g}{r^3} \gtrsim 1 \quad \text{for } \quad r \lesssim (r_c^2 r_g)^{1/3}
\]
Equation of motion

Static, spherically symmetric, non-relativistic source: \( T^\mu_\mu = -\rho \)

\[
\frac{1}{r^2} \left\{ (r^2 \varphi')' + \frac{4r_c^2}{3} [r(\varphi')^2]' \right\} = 8\pi G \rho
\]

Quadratic algebraic equation for \( \varphi' \)

\[
\varphi' = \frac{3r}{8r_c^2} \left( -1 + \sqrt{1 + \frac{16}{3} \left( \frac{r_c^2 r_g}{r^3} \right)} \right)
\]

Vainshtein radius

\[
r_V = (r_c^2 r_g)^{1/3}
\]
Suppose $r_c = 3 \text{ Gpc}$

\[ r_V \sim 100 \text{ pc} \quad \text{for the Sun} \quad (M = M_\odot) \]

\[ r_V \sim 1 \text{ Mpc} \quad \text{for a galaxy cluster} \quad (M = 10^{14} M_\odot) \]
Motivation

- Study the Vainshtein mechanism in the most general scalar-tensor theory, clarifying the conditions under which a screened solution is realized.

- Offer a basic tool to test general scalar-tensor type gravity.
Horndeski’s most general scalar-tensor theory
Galileon

\[ \mathcal{L} = c_1 \phi + c_2 (\partial \phi)^2 + c_3 (\partial \phi)^2 \Box \phi + c_4 (\partial \phi)^2 \left[ (\Box \phi)^2 - (\partial_\mu \partial_\nu \phi)^2 \right] + c_5 (\partial \phi)^2 \left[ (\Box \phi)^3 - 3 \Box \phi (\partial_\mu \partial_\nu \phi)^2 + 2(\partial_\mu \partial_\nu \phi)^3 \right] \]

Vainshtein mechanism operates

Burrage, Seery (2010); ........

Unique scalar-field theory in 4D flat spacetime having

- Galilean shift symmetry \( \phi \rightarrow \phi + b_\mu x^\mu + c \)
- 2nd-order equation of motion
Generalized Galileon

Include gravity

2nd-order equation of motion both for $g_{\mu\nu}$ and $\phi$

Forget about any symmetry...

$$L = K(\phi, X) - G_3(\phi, X) \Box \phi$$

$$+ G_4(\phi, X) R + G_{4X} \left[ (\Box \phi)^2 - (\nabla_\mu \nabla_\nu \phi)^2 \right]$$

$$+ G_5(\phi, X) G_{\mu\nu} \nabla^\mu \nabla^\nu \phi - \frac{1}{6} G_{5X} \left[ (\Box \phi)^3 \right]$$

$$X = -\frac{1}{2} (\partial \phi)^2 - 3 (\Box \phi) (\nabla_\mu \nabla_\nu \phi)^2 + 2 (\nabla_\mu \nabla_\nu \phi)^3$$

$G_{4X} := \frac{\partial G_4}{\partial X}$
Horndeski’s theory

The most general scalar-tensor theory with second-order field equations

\[ \mathcal{L}_H = \delta_{\mu \nu \sigma}^{\alpha \beta \gamma} \left[ \kappa_1 \nabla^\mu \nabla \phi R_{\beta \gamma}^{\nu \sigma} + \frac{2}{3} \kappa_1 X \nabla^\mu \nabla \phi \nabla^\nu \nabla \beta \phi \nabla^\sigma \nabla \gamma \phi 
+ \kappa_3 \nabla \phi \nabla^\mu \phi R_{\beta \gamma}^{\nu \sigma} + 2 \kappa_3 X \nabla \phi \nabla^\mu \phi \nabla^\nu \nabla \beta \phi \nabla^\sigma \nabla \gamma \phi \right] 
+ \delta_{\mu \nu}^{\alpha \beta} \left[ (F + 2W) R_{\alpha \beta}^{\mu \nu} + 2F_X \nabla^\mu \nabla \phi \nabla^\nu \nabla \beta \phi + 2\kappa_8 \nabla \phi \nabla^\mu \phi \nabla^\nu \nabla \beta \phi \right] 
- 6 \left( F_\phi + 2W_\phi - X \kappa_8 \right) \Box \phi + \kappa_9 \]

The generalized Galileon is equivalent to Horndeski’s theory

TK, Yamaguchi, Yokoyama (2011)
Static, spherically symmetric, weak gravitational field

Narikawa, TK, Yamauchi, Saito, 1302.2311
Background

Start with the most general scalar-tensor theory

$$\mathcal{L} = K(\phi, X) - G_3(\phi, X)\Box \phi + G_4(\phi, X)R + G_{4X} [(\Box \phi)^2 - (\nabla_\mu \nabla_\nu \phi)^2]$$

$$+ G_5(\phi, X)G^{\mu\nu} \nabla_\mu \nabla_\nu \phi - \frac{1}{6} G_{5X} [(\Box \phi)^3 - 3 \Box \phi (\nabla_\mu \nabla_\nu \phi)^2 + 2(\nabla_\mu \nabla_\nu \phi)^3]$$

Minkowski background

$$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu, \quad \phi = \phi_0 = \text{const}, \quad X = 0$$

(Require $K(\phi_0, 0) = 0, \quad K_\phi(\phi_0, 0) = 0$ for the theory to admit Minkowski background)
Approximations

Static, spherically symmetric perturbations produced by non-relativistic matter

\[
\begin{align*}
\text{ds}^2 &= -[1 + 2\Phi(r)]dT^2 + [1 - 2\Psi(r)]DX^2 \\
\phi &= \phi_0 + \varphi(r) \\
T_{t}^t &= -\rho(r)
\end{align*}
\]

Perturbations are small, but non-linear terms can be as large as linear terms

Do not neglect \((\partial \partial \epsilon)^n\)

\[
\epsilon \sim \frac{rg}{r} \quad \Rightarrow \quad r_c^2(\partial \partial \epsilon)^2 \gtrsim \partial \partial \epsilon \quad \text{for} \quad r \lesssim (rg r_c^2)^{1/3}
\]

\((\ll 1)\)
Gravitational field equations

Time-time component:

\[
G_4 \frac{(r^2 \Psi')'}{r^2} - G_{4\phi} \frac{(r^2 \phi')'}{2r^2} - (G_{4X} - G_{5\phi}) \frac{[r(\phi')^2]'}{2r^2} + G_{5X} \frac{[(\phi')^3]'}{6r^2} = \frac{\rho}{4}
\]

Space-space component:

\[
2G_4 \frac{[r^2 (\Psi' - \Phi')]'}{r^2} - 2G_{4\phi} \frac{(r^2 \phi')'}{r^2} - (G_{4X} - G_{5\phi}) \frac{[r(\phi')^2]'}{r^2} = 0
\]

Background quantities \( G_{5X} = G_{5X}(\phi_0, 0), \ldots \)

(l.h.s.) = \[
\frac{1}{r^2} \frac{d}{dr} (\cdots)
\]
Scalar field equation

\[
(K_X - 2G_{3\phi}) \left(\frac{r^2 \varphi'}{r^2}\right)' - 2\left(G_{3X} - 3G_{4\phi X}\right) \left[\frac{r(\varphi')^2}{r^2}\right]' \\
+ 2G_{4\phi} \left[\frac{r^2(2\Psi - \Phi)'}{r^2}\right]' + 4\left(G_{4X} - G_{5\phi}\right) \left[\frac{r\varphi'(\Psi' - \Phi')}{r^2}\right]' \\
+ 2\left(G_{4XX} - \frac{2}{3}G_{5\phi X}\right) \left[\left(\frac{\varphi'}{r^2}\right)'\right] + 2G_{5X} \left[\left(\frac{(\varphi')^2}{r^2}\Phi'\right)\right]' \\
= -K_{\phi\phi}\varphi
\]

Neglect “mass term”

(Scalar field is screened if it is sufficiently massive)

\[
\frac{1}{r^2} \frac{d}{dr} (\cdots) = 0
\]
Three equations are integrated once to give algebraic equations for $\Phi'$, $\Psi'$, $\varphi'$

Introduce two mass scales $(M_{\text{Pl}}, \Lambda)$ and six dimensionless parameters

\[
G_4 = \frac{M_{\text{Pl}}^2}{2},
\]
\[
G_{4\phi} = M_{\text{Pl}} \xi,
\]
\[
K_X - 2G_{3\phi} = \eta,
\]
\[
-G_{3X} + 3G_{4\phi X} = \frac{\mu}{\Lambda^3},
\]
\[
G_{4X} - G_{5\phi} = \frac{M_{\text{Pl}}}{\Lambda^3} \alpha,
\]
\[
G_{4XX} - \frac{2}{3}G_{5\phi X} = \frac{\nu}{\Lambda^6},
\]
\[
G_{5X} = -\frac{3M_{\text{Pl}}}{\Lambda^6} \beta
\]

(One is redundant)

Previous example:

$\Lambda \rightarrow \frac{M_{\text{Pl}}}{r_c^2}$
Useful dimensionless quantities:

\[ x(r) := \frac{1}{\Lambda^3} \frac{\varphi'}{r}, \quad A(r) := \frac{1}{M_{Pl}\Lambda^3} \frac{M(r)}{8\pi r^3} \]

Master equations:

\[ \frac{M_{Pl}}{\Lambda^3} \frac{\Phi'}{r} = -\xi x + \beta x^3 + A(r), \]
\[ \frac{M_{Pl}}{\Lambda^3} \frac{\Psi'}{r} = \xi x + \alpha x^2 + \beta x^3 + A(r), \]

and

\[
\begin{align*}
P(x, A) & := \xi A(r) + \left( \frac{\eta}{2} + 3\xi^2 \right) x + \left[ \mu + 6\alpha\xi - 3\beta A(r) \right] x^2 \\
& \quad + \left( \nu + 2\alpha^2 + 4\beta\xi \right) x^3 - 3\beta^2 x^5 \\
& = 0
\end{align*}
\]

Problem reduces to solving **quintic equation**
A(r) \quad \text{(Concrete form depends on density profile)}

Non-linearity sets in, $x(r) \gtrsim 1$, for $A(r) \gtrsim 1$

$A(r_V) = 1$

Linear approximation is good
Solution we are looking for

Asymptotically flat

Outer region

Inner region

Vainshtein mechanism operates

\[ \Phi \sim \Psi \sim \Phi_{GR} \]

\[ x \rightarrow 0 \]

\[ A \ll 1 \iff r \gg r_V \]

\[ A \gg 1 \iff r \ll r_V \]
Outer solution

Linear regime: \[ P(x, A) \approx \xi A(r) + \left( \frac{\eta}{2} + 3\xi^2 \right) x \]

\[ \Leftrightarrow \quad x \approx x_f := -\frac{2\xi A(r)}{\eta + 6\xi^2} \ll 1 \]

Stable if \( \eta + 6\xi^2 > 0 \) (Kinetic term for small fluctuations has right sign)

Other solutions (if they exist) do not correspond to asymptotically flat spacetime
Inner solution

\[ P(x, A) := \xi A(r) + \left( \frac{\eta}{2} + 3\xi^2 \right) x + \left[ \mu + 6\alpha\xi - 3\beta A(r) \right] x^2 + (\nu + 2\alpha^2 + 4\beta\xi) x^3 - 3\beta^2 x^5 = 0 \]

\( \beta = 0 \)

Structure for \( A \gg 1 \) is different depending on whether \( \beta = 0 \) or \( \beta \neq 0 \)

\( P(x, A) \) is cubic — consider separately

\( \beta \neq 0 \)

\[ P(x, A) \approx \xi A - 3\beta Ax^2 - 3\beta^2 x^5 \quad \text{for} \quad A \gg 1 \]
Inner solution for $\beta \neq 0$

$$P(x, A) \approx \xi A - 3\beta Ax^2 - 3\beta^2 x^5$$

- $\xi \beta < 0$
- $x^3 \approx -\frac{A}{\beta}$
- $\frac{\Psi'}{r} \sim A^{2/3}$, $\frac{\Phi'}{r} \sim A^{1/3}$ Not GR

- $\xi \beta > 0$
- $x^3 \approx -\frac{A}{\beta}$ and $x \approx x_\pm := \pm \sqrt[3]{\frac{\xi}{3\beta}}$
- $\Phi \sim \Psi \sim \Phi_{GR}$
Matching inner and outer solutions

(Consider for simplicity the case $\xi > 0$)

$P(x,A)$

Large $|x|$  
Non-linear regime

Small $|x|$  
Linear regime

$\xi A \gg 1$

$\xi A \ll 1$

$x \approx x_f$

inner sol.

outer sol.
Profile of $x$

$x \approx x_f \rightarrow 0$

$A = 1 \iff r = r_V$

$x(A)$

$x \rightarrow x_-$

$A \ll 1$

Outer region

$A \gg 1$

Inner region
Conditions for smooth matching

\[ P(x, A) = 0 \] has a single root in \((x_-, 0)\) for any \(A > 0\)
Otherwise...
Case I

$x_*$ and $A_*$ do not exist satisfying

$$\frac{\partial P(x_*, A_*)}{\partial x} = 0, \quad \frac{\partial^2 P(x_*, A_*)}{\partial x^2} = 0$$

No local extrema in $(x_-, 0)$

$$P(x_-, A) = P(x_-) < 0$$
Case II

\[ \frac{\partial P\left(x_*, A_*\right)}{\partial x} = 0, \quad \frac{\partial^2 P\left(x_*, A_*\right)}{\partial x^2} = 0 \]

Local maximum never exceeds \( P=0 \)
Otherwise...

\[ P(x_*, A_*) > 0 \]
Inner solution for $\beta = 0$

$$P(x, A) \rightarrow \xi A + \left( \frac{\eta}{2} + 3\xi^2 \right) x + (\mu + 6\alpha\xi) x^2 + (\nu + 2\alpha^2) x^3$$

Solution for $A \gg 1$

$$x^3 \approx x_i^3 := -\frac{\xi A}{\nu + 2\alpha^2} \quad (<0)$$

(required from stability)

For this inner solution Vainshtein mechanism operates

$$\Phi \sim \Psi \sim \Phi_{GR}$$
Matching inner and outer solutions

Condition for smooth matching:

no local extrema in $x<0$

Local extrema are in $x>0$ if

$$\mu + 6\alpha \xi < 0$$

No local extrema if

$$(\nu + 2\alpha^2) \ (\eta + 6\xi^2) \geq \frac{2}{3} \ (\mu + 6\alpha \xi)^2$$

$$\mu + 6\alpha \xi \geq 0$$

$$y = -\left(\frac{\eta}{2} + 3\xi^2\right) x - (\mu + 6\alpha \xi) x^2 - (\nu + 2\alpha^2) x^3$$
Decoupling limit of massive gravity  

\[ L = -\frac{1}{2} h^{\mu\nu} e_{\mu\nu}^{\alpha\beta} h_{\alpha\beta} + h^{\mu\nu} \left( X^{(1)}_{\mu\nu} + \frac{a_2}{\Lambda^3} X^{(2)}_{\mu\nu} + \frac{a_3}{\Lambda^6} X^{(3)}_{\mu\nu} \right) + \frac{1}{2M_{Pl}} h^{\mu\nu} T_{\mu\nu} \]

Helicity-2 mode \quad Interactions with helicity-0 mode

\[ K = 0 = G_3 \]
\[ G_4 = \frac{M_{Pl}^2}{2} + M_{Pl} \phi + \frac{M_{Pl}}{\Lambda^3} \alpha X \]

"Covariantization" \quad de Rham, Eisenberg (2011)

\[ G_5 = -3 \frac{M_{Pl}}{\Lambda^6} \beta X \]

in Horndeski's language

- \( M_{Pl} \to \infty, \ m \to 0 \)
- \( \Lambda^3 = M_{Pl} m^2 \) fixed
Decoupling limit of massive gravity = 2-parameter subclass of Horndeski’s theory

Smooth matching of asymptotically flat and Vainshtein solutions is possible for:

\[ \beta = \left( \frac{5 + \sqrt{13}}{24} \right) \alpha^2 \]

Previous results [Sjors and Mortsell (2011); Sbisa et al. (2012)] are correctly reproduced.
Application
Application: Gravitational lensing

Lensing convergence can be computed for any density profile and for any scalar-tensor theory from

\[ \Delta(\Phi + \Psi) = \frac{\Lambda^3}{M_{\text{Pl}}} \frac{1}{r^2} \frac{d}{dr} \left[ r^2(\alpha x^2 + 2\beta x^3 + 2A) \right] \cap x'(r) \]

Convergence:

\[ \kappa(\theta) = \frac{(\chi_S - \chi_L)\chi_L}{\chi_S} \int_0^\infty dZ \frac{\Delta}{a_L^2}(\Phi + \Psi) \]

Interesting signature in cluster lensing?
$x'(r)$ can be large at transition from screened to unscreened regions.

![Graphs showing $x(r)$ and $x'(r)$ for different values of $\alpha$ and $\beta$.]

Sharp transition occurs for parameters near boundary.
Dip in convergence power spectrum
(if we are lucky enough…?)

\[
\kappa(\theta)
\]

\[
\theta \text{ [arcmin]}
\]

\[
\alpha=0.5, \beta=0.3 \quad \alpha=0.8, \beta=0.34 \quad \alpha=0.985, \beta=0.375 \quad \Lambda\text{CDM}
\]

\[
\Lambda^3=\left(100H_0\right)^2M_{\text{Pl}}
\]
Dip is not a consequence of the specific choice of the density profile.
Static, spherically symmetric, weak gravitational field sourced by non-relativistic matter in **Horndeski’s most general scalar-tensor theory**

The problem reduces to solving a **quintic algebraic equation**

Conditions under which a screened solution is realized are clarified

Interesting applications such as testing gravity with cluster lensing

Cosmological background? —— Sixth-order algebraic equation with time-dependent coefficients… [Kimura, TK, Yamamoto (2012)]

Application to other cosmological probes?
Thank you!