Calculating one loop amplitudes efficiently

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Introduction

- Fully automated cross section calculations for LHC in SM and BSM
  ALPGEN, MADGRAPH,COMIX,... $d\sigma_n^{(0)} \approx |M_n^{(0)}|^2 d\Phi_{n-2}$, n=3...12,...

- At LHC tree level is not enough, NLO precision is mandatory

Federix

$$d\sigma_n^{(1)} \approx |M_n^{(0)}|^2 d\Phi_{n-2} + 2\mathrm{Re}(M_n^{(0)\dagger} M_n^{(1)}) d\Phi_{n-2} + |M_{n+1}^{(0)}|^2 d\Phi_{n-1}$$

this talk

subtracted

- **Color-dressed formulation** of the D-dimensional generalized unitarity and testing its efficiency by calculating $M_n^{(1)}$, n=4,.....12 (gluons)

- Recursive calculation of three amplitudes with slight deformation of the momenta of two external legs (complex, D-dimensional, also for spin)

- Calculation of loop amplitudes from tree amplitudes

The Unitarity Method: efficient P(E)-algorithm at NLO

Gauge theory one-loop amplitudes from tree amplitudes many analytic results, with applications to collider physics Bern,Dixon, Kosower, 1994-2007,...

Generalized cuts, box coefficients Britto, Cachazo, Feng 2004
Recursion relations for the rational parts Beger et.al. 2005

Parametric integration method of Ossola, Papadopoulos, Pittau (2006)
Numerical algorithmic implementation of generalized unitarity Ellis, Giele, ZK (2007)
Generalized D-dimensional unitarity Giele, ZK, Melnikov (2008),
Numerical algorithmic implementation by Black Hat Collaboration (2008)

Explicit NLO generators for calculating NLO cross-sections of 6-leg process for LHC and Tevatron Ellis,Melnikov,Zanderighi; BlackHatCollaboration; van Hameren et.al., Bevilaqua (2009)

Complementary analytic methods Badger (2008), Mastrolia (2009)
Independent numerical implementations (Giele, Winter (2008), Lazoploulos (2008), etc)

D-Dimensional unitarity method

for color and flavor ordered primitive amplitude

$$\mathcal{A}_N(p_1,p_2,\ldots,p_N;l) = \frac{\mathcal{N}(p_1,p_2,\ldots,p_N;l)}{d_1 d_2 \cdots d_N} = \sum_{i_1\le i_2\cdots\le i_5\le N} \frac{e_{i_1 i_2 i_3 i_4 i_5}}{d_{i_1} d_{i_2} d_{i_3} d_{i_4} d_{i_5}} +$$

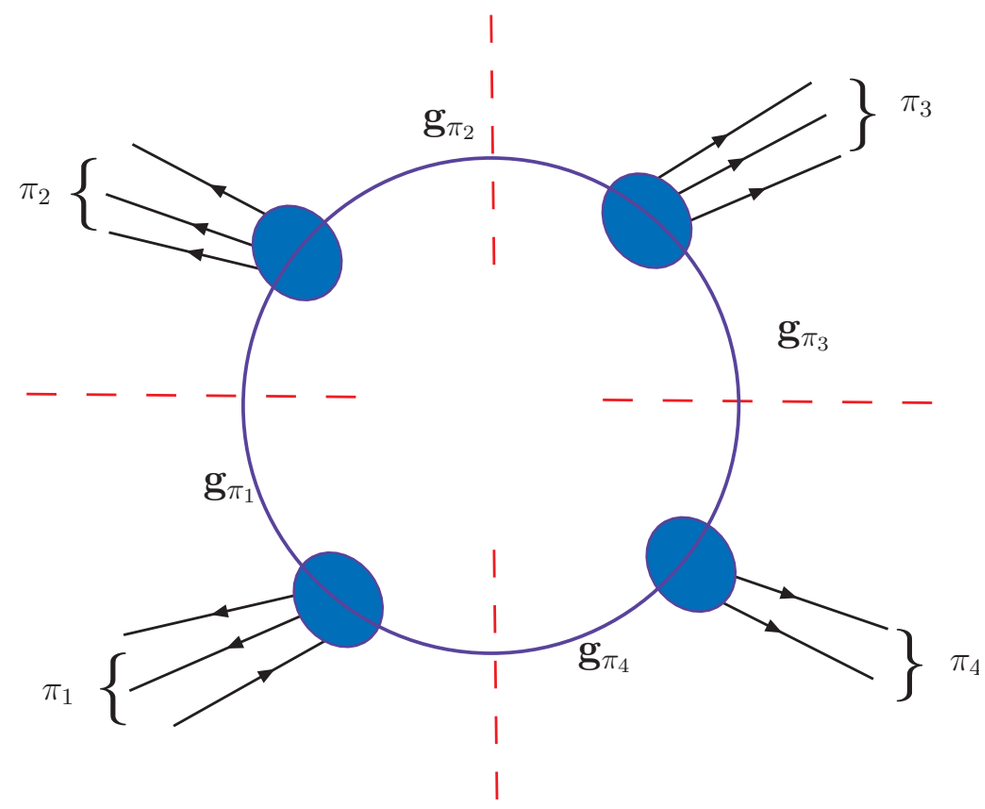
$$\sum_{1\le i_1<i_2<i_3<i_4\le N} \frac{\overline{d}_{i_1 i_2 i_3 i_4}(l)}{d_{i_1} d_{i_2} d_{i_3} d_{i_4}} + \sum_{1\le i_1<i_2<i_3\le N} \frac{\overline{c}_{i_1 i_2 i_3}(l)}{d_{i_1} d_{i_2} d_{i_3}} + \sum_{1\le i_1<i_2\le N} \frac{\overline{b}_{i_1 i_2}(l)}{d_{i_1} d_{i_2}} + \sum_{1\le i_1\le N} \frac{\overline{a}_{i_1}(l)}{d_{i_1}}$$

$$\mathcal{N}^{(D_s)}(l) = \mathcal{N}_0(l) + (D_s-4)\mathcal{N}_1(l)$$

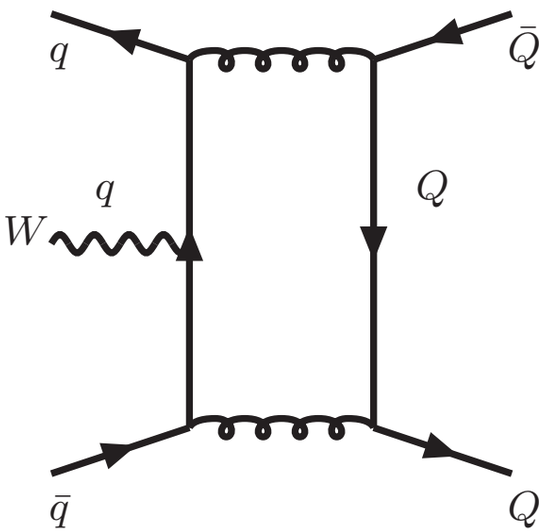
$$\mathrm{Res}_{ij\cdots k}\left[F(l)\right] \equiv \Big[d_i(l) d_j(l) \cdots d_k(l) F(l)\Big]\Big|_{l=l_{ij\cdots k}} .$$

$$\overline{d}_{ijkl}(l) = \mathrm{Res}_{ijkl}\Big(\mathcal{A}_N(l)\Big) = \sum A_i \times A_j \times A_k \times A_l$$

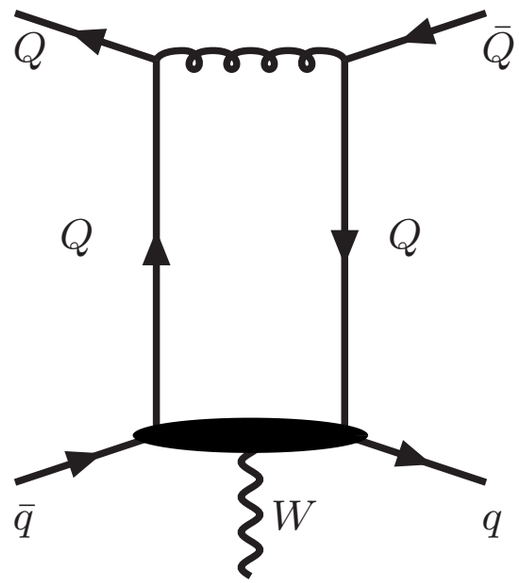
$$\overline{c}_{ijk}(l) = \mathrm{Res}_{ijk}\left(\mathcal{A}_N(l) - \sum_{l\neq i,j,k} \frac{\overline{d}_{ijkl}(l)}{d_i d_j d_k d_l}\right)$$



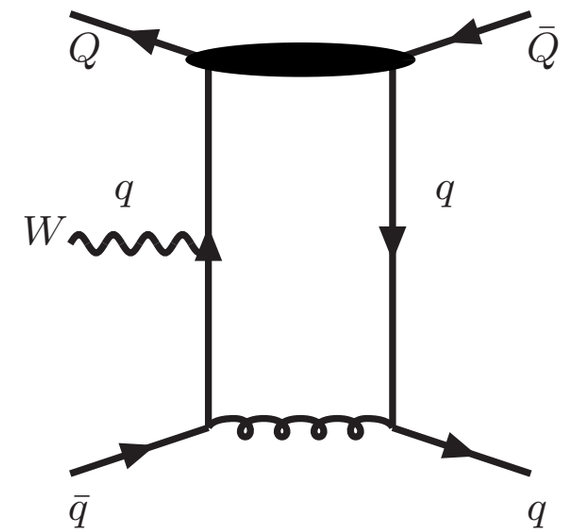
Decomposition to color and flavor ordered primitive amplitudes is cumbersone



$a)$



$c)$



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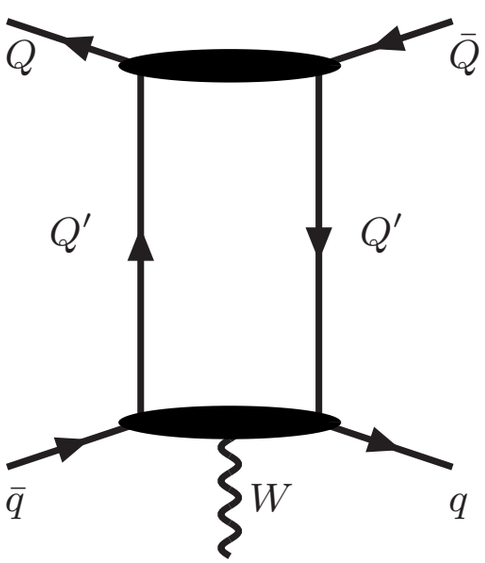


proportional to n_f

Tree level: BG and BCFW recursion relations for colorless ordered amplitudes different color basis (T-basis, F-basis, mixed basis, color-flow basis)

One loop: Decomposition to color and flavor ordered primitive amplitudes

Difficulty: It becomes cumbersome for increasing number of flavor. No color ordering in QED, electroweak.

All use color ordered and primitive amplitudes

$$\mathcal{B}^{\text{1-loop}}(1_{\bar{q}},2_q,3_{\bar{Q}},4_Q,5_g) = g^5\Big[N_c(T^{a_5})_{i_4}^{\ \bar{\imath}_1}\delta_{i_2}^{\ \bar{\imath}_3}\boxed{B_{7;1}} + (T^{a_5})_{i_2}^{\ \bar{\imath}_1}\delta_{i_4}^{\ \bar{\imath}_3}B_{7;2} + N_c(T^{a_5})_{i_2}^{\ \bar{\imath}_3}\delta_{i_4}^{\ \bar{\imath}_1}\boxed{B_{7;3}} + (T^{a_5})_{i_4}^{\ \bar{\imath}_3}\delta_{i_2}^{\ \bar{\imath}_1}B_{7;4}\Big].$$

$$\begin{aligned} B_{7;1}^{[1],a} = {} & \left(1-\frac{1}{N_c^2}\right)A_L^{[1],a}(1_{\bar{q}},2_q,3_{\bar{Q}},4_Q,5_g) - \frac{1}{N_c^2}\Big(-A_L^{[1],a}(1_{\bar{q}},5_g,2_q,3_{\bar{Q}},4_Q) \\ & -A_L^{[1],a}(1_{\bar{q}},5_g,2_q,4_Q,3_{\bar{Q}}) - A_L^{[1],a}(1_{\bar{q}},2_q,5_g,3_{\bar{Q}},4_Q) - A_L^{[1],a}(1_{\bar{q}},2_q,5_g,4_Q,3_{\bar{Q}}) \\ & -A_L^{[1],a}(1_{\bar{q}},2_q,3_{\bar{Q}},5_g,4_Q) - A_L^{[1],a}(1_{\bar{q}},2_q,4_Q,5_g,3_{\bar{Q}}) + A_L^{[1],a}(1_{\bar{q}},2_q,4_Q,3_{\bar{Q}},5_g)\Big), \end{aligned} \tag{5.6}$$

$$\begin{aligned} B_{7;2}^{[1],a} = {} & +A_L^{[1],a}(1_{\bar{q}},2_q,5_g,4_Q,3_{\bar{Q}}) - A_L^{[1],a}(1_{\bar{q}},2_q,4_Q,5_g,3_{\bar{Q}}) + A_L^{[1],a}(1_{\bar{q}},2_q,4_Q,3_{\bar{Q}},5_g) \\ & -\frac{1}{N_c^2}\Big(A_L^{[1],a}(1_{\bar{q}},5_g,2_q,3_{\bar{Q}},4_Q) + A_L^{[1],a}(1_{\bar{q}},5_g,2_q,4_Q,3_{\bar{Q}})\Big), \end{aligned} \tag{5.7}$$

$$\begin{aligned} B_{7;3}^{[1],a} = {} & \left(1-\frac{1}{N_c^2}\right)A_L^{[1],a}(1_{\bar{q}},2_q,5_g,3_{\bar{Q}},4_Q) - \frac{1}{N_c^2}\Big(-A_L^{[1],a}(1_{\bar{q}},5_g,2_q,3_{\bar{Q}},4_Q) \\ & -A_L^{[1],a}(1_{\bar{q}},5_g,2_q,4_Q,3_{\bar{Q}}) - A_L^{[1],a}(1_{\bar{q}},2_q,3_{\bar{Q}},5_g,4_Q) - A_L^{[1],a}(1_{\bar{q}},2_q,4_Q,5_g,3_{\bar{Q}}) \\ & -A_L^{[1],a}(1_{\bar{q}},2_q,3_{\bar{Q}},4_Q,5_g) - A_L^{[1],a}(1_{\bar{q}},2_q,4_Q,3_{\bar{Q}},5_g) + A_L^{[1],a}(1_{\bar{q}},2_q,5_g,4_Q,3_{\bar{Q}})\Big), \end{aligned} \tag{5.8}$$

Dressed recursive technique for tree amplitudes

Draggiotis, Kleiss, Papadopoulos, Duhr, Maltoni, Comix:Gleisberg, Hoche

Monte-Carlo sampling over flavor, helicity, color, momentum quantum numbers of external sources

$$\mathbf{f}_i^{(r)} = \{f_i, h_{f_i}, C_{f_i}, K_i\}^{(r)} \quad ,$$

$$d\,\sigma_{LO}(f_1 f_2 \to f_3 \cdots f_n) = \frac{W_S}{N_{\rm event}} \times \sum_{r=1}^{N_{\rm event}} d\,PS^{(r)}(K_1 K_2 \to K_3 \cdots K_n) \left| \mathcal{M}^{(0)}\left(\mathbf{f}_1^{(r)}, \mathbf{f}_2^{(r)}, \ldots, \mathbf{f}_n^{(r)}\right) \right|^2$$

$$\mathcal{M}^{(0)}\big(\mathbf{f}_1, \ldots, \mathbf{f}_n\big) \;=\; P^{-1}\left[J\big(\mathbf{f}_1, \ldots, \mathbf{f}_{n-1}\big), J\big(\mathbf{f}_n\big)\right] \;.$$

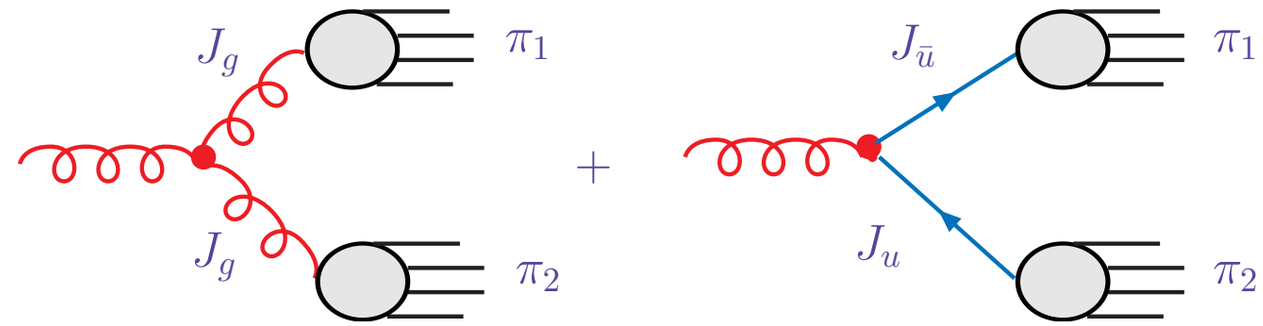
Recursion relations are defined in terms of currents with n-1 on-shell external legs and one off-shell leg

$$P\left[J(\{\mathbf{f}\}_{\pi_1}), J(\{\mathbf{f}\}_{\pi_2})\right] \;=\; \sum_{\mathbf{g_1 g_2}} J_{\mathbf{g_1}}(\{\mathbf{f}\}_{\pi_1}) P^{\mathbf{g_1 g_2}}(K_{\pi_1}) J_{\mathbf{g_2}}(\{\mathbf{f}\}_{\pi_2})$$

$$\mathbf{g} = \{g, L_g, C_g, K_g\}$$

Generic recursion with generic vertices for tree amplitudes

$D_{\rm gluon}(\pi_1,\pi_2) \;=$



$+\dots$

Vertices:

$$D_{\mathbf{g_1}\cdots\mathbf{g_k}}(Q_1,\dots,Q_k) = D^{L_{g_1}\cdots L_{g_k}}_{g_1,\dots g_k;C_{g_1}\cdots,C_{g_k}}(Q_1,\dots,Q_k)\ .$$

$$\boxed{D_{\mathbf{g}}\left[J(\mathbf{f}(\pi_1)),\dots,J(\mathbf{f}(\pi_k))\right] = \sum_{\mathbf{g_1}\cdots\mathbf{g_k}} D_{\mathbf{g g_1}\cdots\mathbf{g_k}}(-K_{\Pi_k},K_{\pi_1},\dots,K_{\pi_k})\times J^{\mathbf{g_1}}(\mathbf{f}_{\pi_1})\times\cdots\times J^{\mathbf{g_k}}(\mathbf{f}_{\pi_k})\ ,}$$

$\Pi_k = \bigcup_{i=1}^k \pi_i$

BG recursion relations:

$$J_{\mathbf{g}}(\mathbf{f}_i) \;=\; \delta^{g f_i}\delta^{C_g C_{f_i}}\, S^{h_{f_i} C_{f_i}}_{f_i L_g}(K_i)\ .$$

Stirling number of 2nd kind

$$\boxed{J_{\mathbf{g}}(\mathbf{f}_1,\dots,\mathbf{f}_n) \;=\; \sum_{k=2}^{V_{\max}-1}\ \sum_{P_{\pi_1\cdots\pi_k}(1,\dots,n)}^{\mathcal{S}_2(n,k)} P_{\mathbf{g}}\Big[D\big[J(\mathbf{f}_{\pi_1}),\dots,J(\mathbf{f}_{\pi_k})\big]\Big]\ ,}$$

all possible partions of n particle into k subsets is given by the Stirling number of second kind $S_2(n,k)$.

Color-flow notation

$$|\mathcal{M}|^2 = \mathcal{M}^a\,\delta_{ab}\,\left(\mathcal{M}^b\right)^\dagger = \mathcal{M}^a\,T^a_{ij}T^b_{ji}\,\left(\mathcal{M}^b\right)^\dagger = \mathcal{M}_{ij}\,\mathcal{M}^\dagger_{ji}\ .$$

Vertices

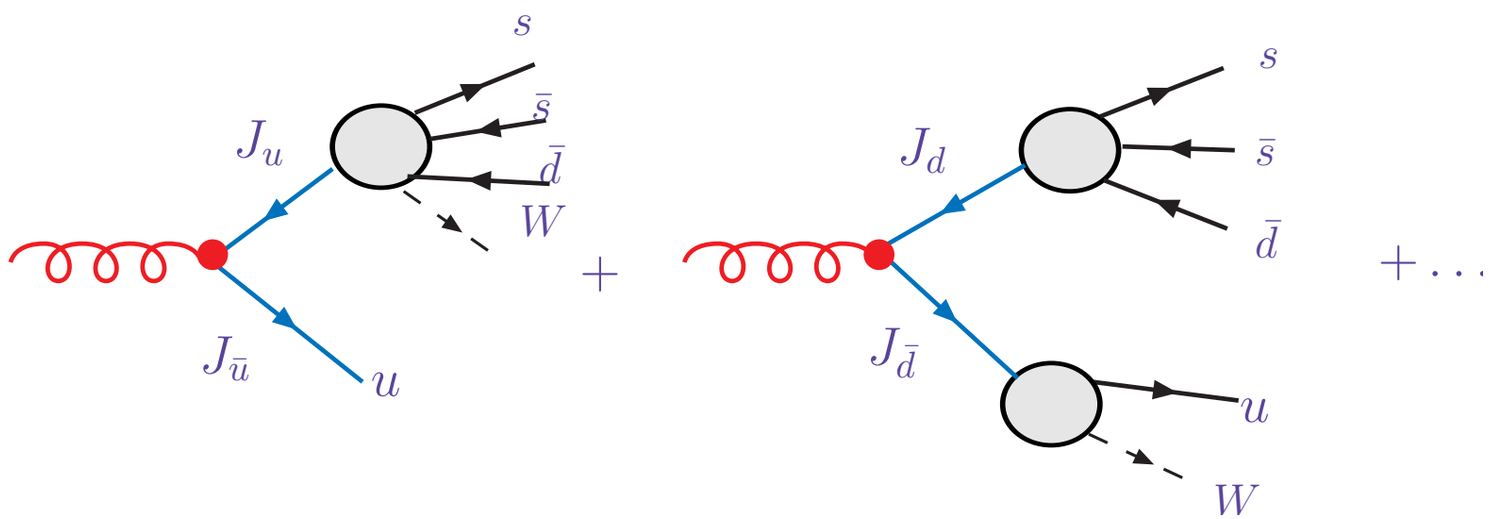
$$D_{\mathbf{g}_1\mathbf{g}_2\mathbf{g}_3}(K_1,K_2,K_3) \quad = \quad \left(\delta^{i_1}_{j_2}\delta^{i_2}_{j_3}\delta^{i_3}_{j_1} - \delta^{i_1}_{j_3}\delta^{i_2}_{j_1}\delta^{i_3}_{j_2}\right)\widehat{V}_3^{\mu_1\mu_2\mu_3}(K_1,K_2,K_3)\ ,$$

$$\widehat{V}_3^{\mu_1\mu_2\mu_3}(K_1,K_2,K_3) = \frac{1}{\sqrt{2}}\left((K_1-K_2)^{\mu_3}g^{\mu_1\mu_2} + (K_2-K_3)^{\mu_1}g^{\mu_2\mu_3} + (K_3-K_1)^{\mu_2}g^{\mu_3\mu_1}\right)\ .$$

$$D_{\mathbf{g}_1\mathbf{g}_2\mathbf{g}_3\mathbf{g}_4} = D^{\mu_1\mu_2\mu_3\mu_4}_{i_1j_1i_2j_2i_3j_3i_4j_4} = \sum_{C(234)}\left(\delta^{i_1}_{j_2}\delta^{i_2}_{j_3}\delta^{i_3}_{j_4}\delta^{i_4}_{j_1} + \delta^{i_1}_{j_4}\delta^{i_2}_{j_1}\delta^{i_3}_{j_2}\delta^{i_4}_{j_3}\right)\widehat{V}_4^{\mu_1\mu_3;\mu_2\mu_4}\ ,$$

BG recursion relations:

$J_g(u, \bar{d}, s, \bar{s}, W) =$



$$
\begin{aligned}
P_{\mathbf{g}}\left[D\left[J(\mathbf{f}_{\pi_1}), J(\mathbf{f}_{\pi_2})\right]\right] &= \frac{1}{K^2_{\Pi_2}} D^{i;IJ;j}_{s_1 \mu s_2} \times J^i_{s_1}(\mathbf{f}_{\pi_1}) \times J^j_{s_2}(\mathbf{f}_{\pi_2}) \\
&+ \frac{1}{K^2_{\Pi_2}} D^{IJ;i_1 j_1;i_2 j_2}_{\mu \mu_1 \mu_2}(-K_{\pi_1 \cup \pi_2}, K_{\pi_1}, K_{\pi_2}) \times J^{\mu_1}_{i_1 j_1}(\mathbf{f}_{\pi_1}) \times J^{\mu_2}_{i_2 j_2}(\mathbf{f}_{\pi_2}) \, .
\end{aligned}
$$

Color sampling for n-gluon scattering

The full color configuration of the event is expressed by $\{(ij)_m\}_{m=1}^n$ where i_m and j_m each denote a color state out of three possible ones that can be labeled $\{1,2,3\}$.

Naive:

$$N_{\rm col}^{\rm Naive} = 9^n \qquad\qquad W_{\rm col}^{\rm Naive} = 1$$

$$\exists\, c \in \{1,2,3\} : \sum_{m=1}^n \left(\delta_{i_m,c} - \delta_{j_m,c}\right) \neq 0$$

Conserved:

$$N_{\rm col}^{\rm Conserved} = \sum_{n_1,n_2,n_3=0}^{n} \delta_{n_1+n_2+n_3,n} \left(\frac{n!}{n_1!\,n_2!\,n_3!}\right)^2$$

$$W_{\rm col}^{\rm Conserved} = 3^n \frac{n!}{n_1!\,n_2!\,n_3!}\,.$$

Non-zero:

Color sampling for n-gluon scattering

Scattering	Naive	Conserved	Non-Zero
$2 \to 2$	6,561	639	378
$2 \to 3$	59,049	4,653	3,180
$2 \to 4$	531,441	35,169	27,240
$2 \to 5$	4,782,969	272,835	231,672
$2 \to 6$	43,046,721	2,157,759	1,949,178
$2 \to 7$	387,420,489	17,319,837	16,279,212
$2 \to 8$	3,486,784,401	140,668,065	135,526,716

Time in secs to evaluate 10000 color-dressed tree amplitude

Scaling:

$$T_n = \sum_{m=2}^{n-1} \binom{n-1}{m} \mathcal{S}_2(m, V-1) = \mathcal{S}_2(n, V) \sim V^n$$

Scattering	color ordered	color dressed ($V_{\max} = 4$)	color dressed ($V_{\max} = 3$)
$2 \to 2$	0.0313	0.117	0.083
$2 \to 3$	0.169 (5.40)	0.495 (4.24)	0.327 (3.93)
$2 \to 4$	0.791 (4.68)	1.556 (3.14)	0.822 (2.51)
$2 \to 5$	3.706 (4.69)	6.11 (3.93)	2.66 (3.23)
$2 \to 6$	17.83 (4.81)	25.26 (4.13)	7.55 (2.84)
$2 \to 7$	99.79 (5.60)	93.43 (3.70)	24.9 (3.30)
$2 \to 8$	557.9 (5.59)	392.4 (4.20)	76.1 (3.05)
$2 \to 9$	2,979 (5.34)	1,528 (3.89)	228 (2.99)
$2 \to 10$	19,506 (6.55)	5,996 (3.92)	693 (3.04)
$2 \to 11$	118,635 (6.08)	24,821 (4.14)	
$\vdots$		$\vdots$	
$2 \to 15$		6,248,300 (3.98^4)	

Dressed recursive technique for one-loop amplitudes

Monte-Carlo sampling over quantum numbers of external sources

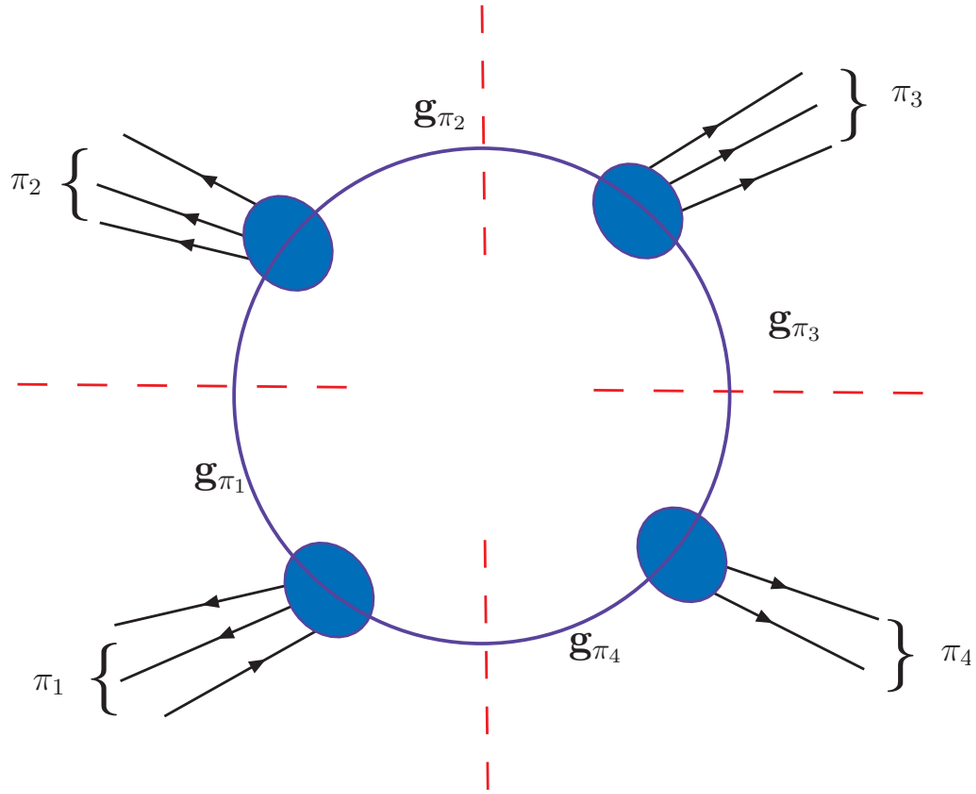
$$d\,\sigma^{(V)}(f_1 f_2 \to f_3 \cdots f_n) = \frac{W_S}{N_{\rm event}} \times \sum_{k=1}^{N_{\rm event}} d\,PS^{(k)}(K_1 K_2 \to K_3 \cdots K_n)$$
$$2\Re\left(\mathcal{M}^{(0)}\left(\mathbf{f}_1^{(k)},\ldots,\mathbf{f}_n^{(k)}\right)^\dagger \times \mathcal{M}^{(1)}\left(\mathbf{f}_1^{(k)},\ldots,\mathbf{f}_n^{(k)}\right)\right)$$

$$\mathcal{M}^{(1)}\left(\mathbf{f}_1,\ldots,\mathbf{f}_n\right) = \int \frac{d^D\ell}{(2\pi)^D}\,\mathcal{A}^{(1)}\left(\mathbf{f}_1,\ldots,\mathbf{f}_n \mid \ell\right)$$

Generalized OPP parametrization

$$\mathcal{A}^{(1)}\left(\mathbf{f}_1,\ldots,\mathbf{f}_n \mid \ell\right) = \sum_{k=1}^{C_{\max}} \sum_{RP_{\pi_1\cdots\pi_k}(1,2,\ldots,n)}^{\max\left(1,\frac{1}{2}(k-1)!\right)\mathcal{S}_2(n,k)} \sum_{g_{\Pi_1},\ldots,g_{\Pi_k}} \frac{\mathcal{P}_k\left(\vec{C}_{g_{\Pi_1}\cdots g_{\Pi_k}} \mid \ell\right)}{d_{g_{\Pi_1}}(\ell)\,d_{g_{\Pi_2}}(\ell)\cdots d_{g_{\Pi_k}}(\ell)}\,,$$

$$\Pi_k = \bigcup_{i=1}^k \pi_i$$



- **Sum over ordered cuts changes to sum over partitions including non-cyclic, non-reflective premutations of the external partons**

$$\mathcal{A}^{(1)}\left(\mathbf{f}_1,\ldots,\mathbf{f}_n \mid \ell\right) = \sum_{k=1}^{C_{\max}} \sum_{RP_{\pi_1\cdots\pi_k}(1,2,\ldots,n)}^{\max\left(1,\frac{1}{2}(k-1)!\right)\mathcal{S}_2(n,k)} \sum_{g_{\Pi_1},\ldots,g_{\Pi_k}} \frac{\mathcal{P}_k\left(\vec{C}_{g_{\Pi_1}\cdots g_{\Pi_k}} \mid \ell\right)}{d_{g_{\Pi_1}}(\ell)\, d_{g_{\Pi_2}}(\ell)\cdots d_{g_{\Pi_k}}(\ell)} ,$$

- sum over the propagator flavors $g_{\Pi_1},\ldots,g_{\Pi_k}$ is required as these are not uniquely defined for unordered amplitudes

Sum over ordered cuts changes to sum over partitions including non-cyclic, non-reflective premutations of the external partons

Number of the cuts is higher.

$$\mathcal{A}^{(1)}\left(\mathbf{f}_1,\ldots,\mathbf{f}_n \mid \ell\right) = \sum_{k=1}^{C_{\max}} \sum_{RP_{\pi_1\cdots\pi_k}(1,2,\ldots,n)}^{\max\left(1,\frac{1}{2}(k-1)!\right)\mathcal{S}_2(n,k)} \sum_{g_{\Pi_1},\ldots,g_{\Pi_k}} \frac{\mathcal{P}_k\left(\vec{C}_{g_{\Pi_1}\cdots g_{\Pi_k}} \mid \ell\right)}{d_{g_{\Pi_1}}(\ell)\, d_{g_{\Pi_2}}(\ell)\cdots d_{g_{\Pi_k}}(\ell)} ,$$

Unordered cuts.

#	pentagon	box	triangle	bubble	sum	$\frac{\text{sum}_n}{\text{sum}_{n-1}}$	ordr total/unordr total		
n	cuts	cuts	cuts	cuts	$\equiv$ total		orderings	$(ab)_k$	$(cd)_k$
4	0	3	6	3	12		2.750	1.833	2.750
5	12	30	25	10	77	6.42	4.052	2.026	2.364
6	180	195	90	25	490	6.36	6.857	2.743	2.514
7	1,680	1,050	301	56	3,087	6.30	13.06	4.354	1.451
8	12,600	5,103	966	119	18,788	6.09	28.17	8.048	1.610
9	83,412	23,310	3,025	246	109,993	5.85	68.18	17.05	2.557
10	510,300	102,315	9,330	501	622,446	5.66	182.8	40.61	2.708
11	2,960,760	437,250	28,501	1,012	3,427,523	5.51	535.7	107.1	
12	16,552,800	1,834,503	86,526	2,035	18,475,864	5.39	1699	308.9	

$$\frac{(n-1)!}{2} \to S_2(n,5) \approx 5^n$$

C++ code for dressed amplitudes

- Method is based on :

- Ellis, Giele, KZ, arXiv:0708.2398, 4D method, cut constructible part
- Giele, Melnikov, KZ. arXiv:0801.2237, arXiv:806.3467, D-Dim. method, full amplitude
- Giele, Zanderighi, arXiv:0805.2152, application of DD method to pure gluons

- Implemented algorithm based on previous implementation of ordered amplitudes Giele and Winter, arXiv:0902.0094

- New features:

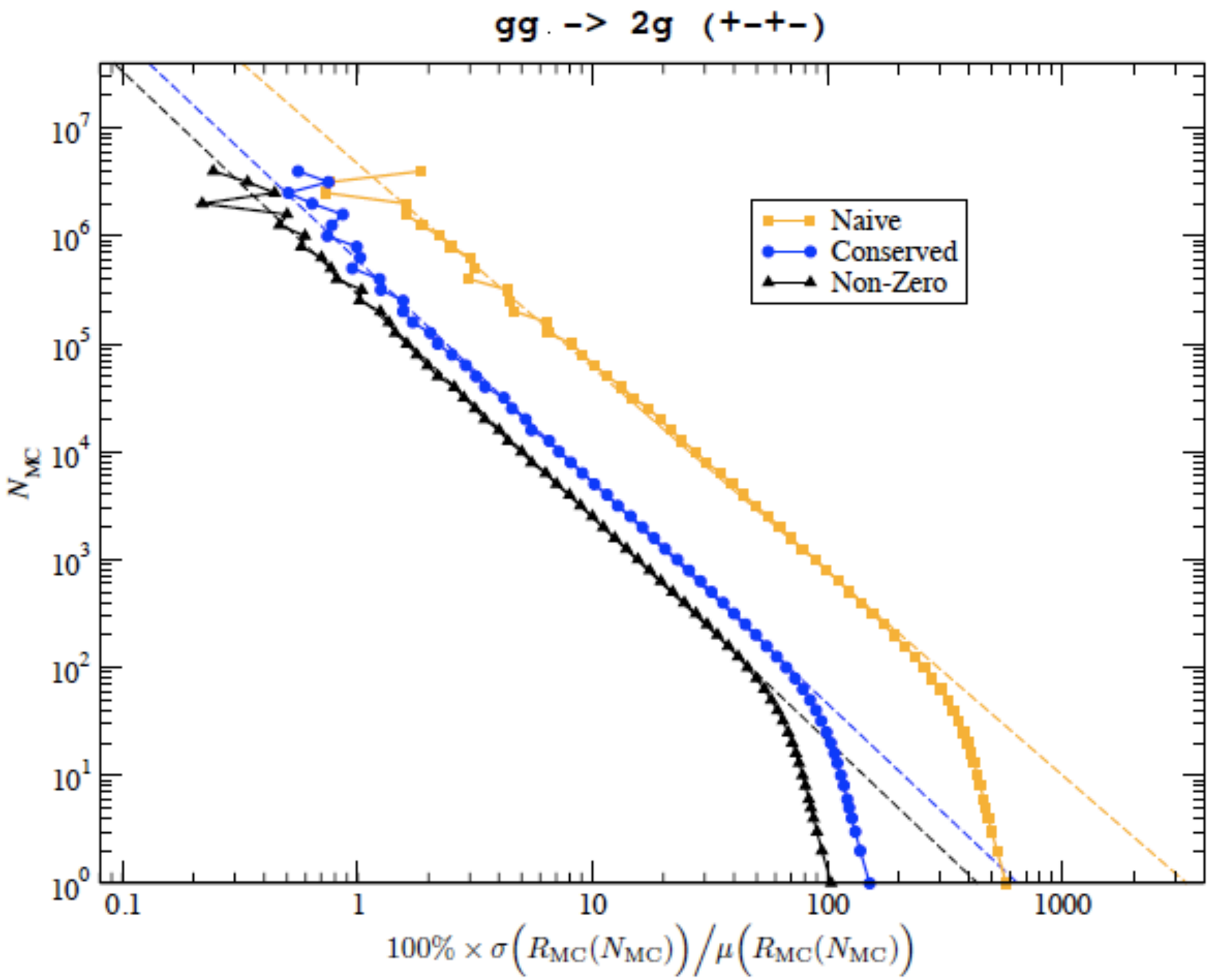
- Sum over ordered cuts changes to sum over partitions including non-cyclic, non-reflective premutations of the external partons

$$\sum_{[i_1|i_k]} \rightarrow \sum_{RP_{\pi_1\ldots\pi_k}(1,2,..,N)}$$

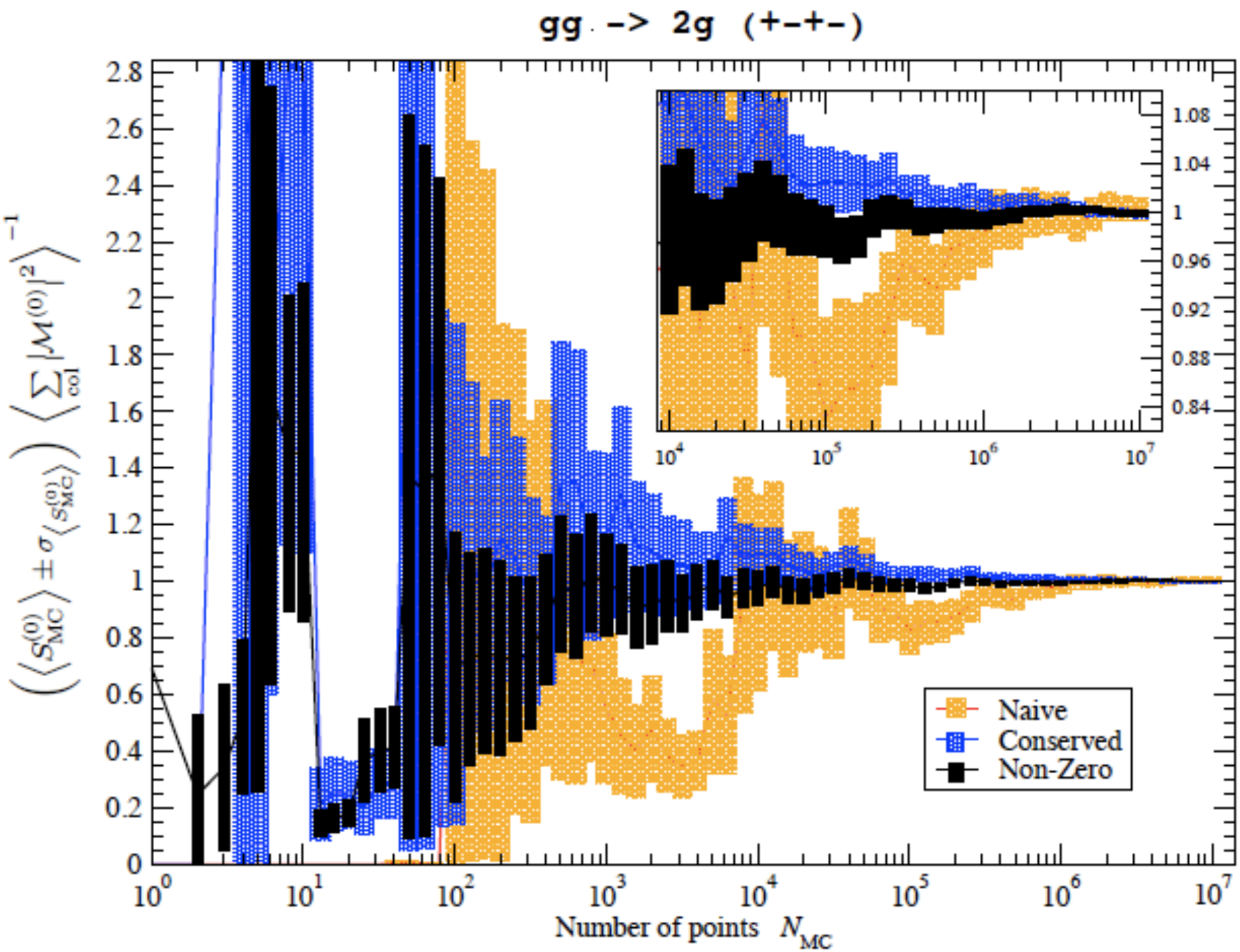
- Residues are given by color dressed tree amplitudes and summed over internal color and spin

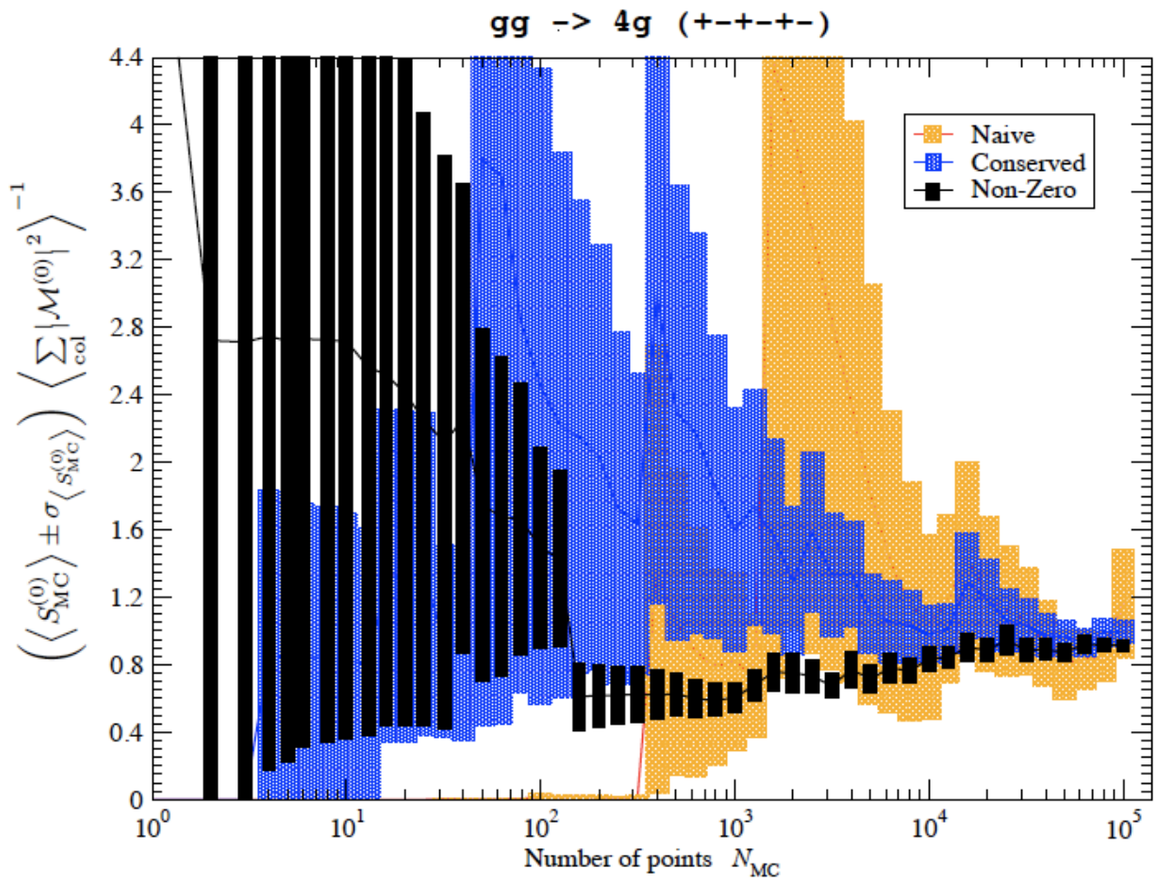
- Symmetry factor, e.g. 1/2 for bubble

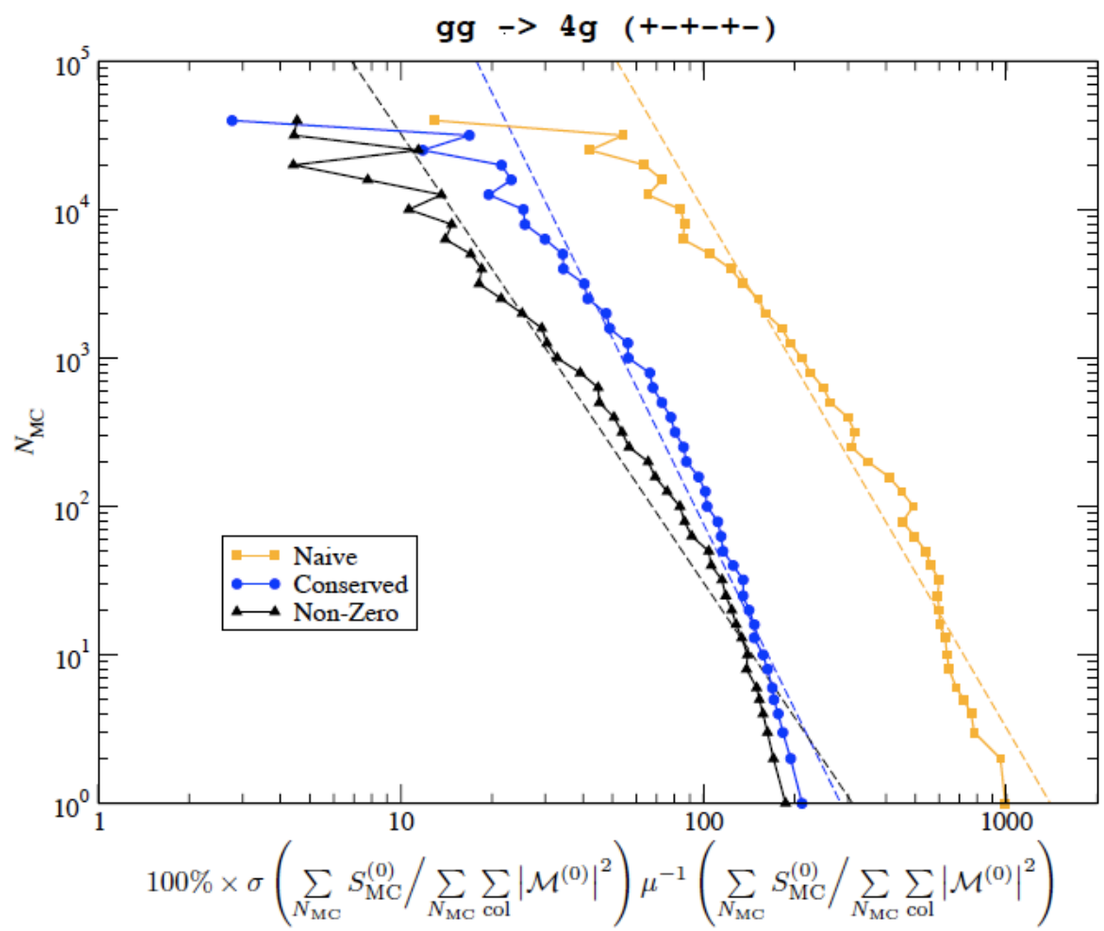
$$S^{(0)}_{\mathrm{MC}} \;=\; W_{\mathrm{col}}(n_1, n_2, n_3) \times \left| \mathcal{M}^{(0)}(\mathbf{g}_1, \ldots, \mathbf{g}_n) \right|^2 \;,$$



$$S_{\mathrm{MC}}^{(0)} = W_{\mathrm{col}}(n_1, n_2, n_3) \times \left| \mathcal{M}^{(0)}(\mathbf{g}_1, \ldots, \mathbf{g}_n) \right|^2 ,$$



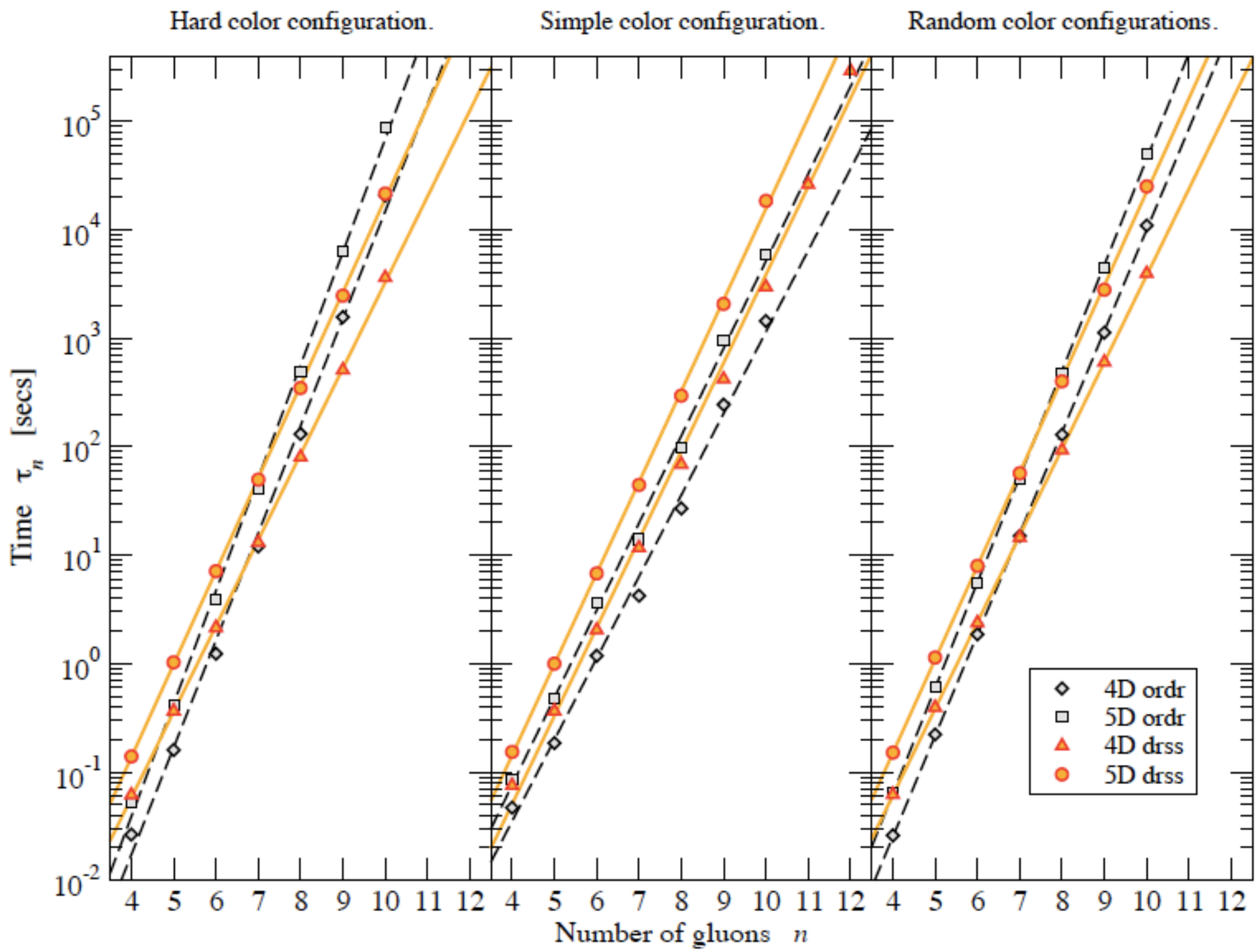

gg -> 4g (+-+-+-)
Naive
Conserved
Non-Zero
Number of points N_{MC}


gg -> 4g (+-+-+-)
Naive
Conserved
Non-Zero
N_{MC}

Numerical results for loop amplitudes (n-gluons)

n	4D-case							5D-case						
	ordr			drss			$\frac{\text{ordr}}{\text{drss}}$	ordr			drss			$\frac{\text{ordr}}{\text{drss}}$
	$\tau_n^{(a)}$	$\tau_n^{(b)}$	r_n	$\tau_n^{(a)}$	$\tau_n^{(b)}$	r_n		$\tau_n^{(a)}$	$\tau_n^{(b)}$	r_n	$\tau_n^{(a)}$	$\tau_n^{(b)}$	r_n	
4	0.027	0.026		0.061	0.062		0.43	0.053	0.052		0.139	0.140		0.38
5	0.159	0.161	6.04	0.368	0.364	5.95	0.44	0.415	0.412	7.88	1.026	1.029	7.37	0.40
6	1.234	1.235	7.72	2.152	2.146	5.87	0.57	3.887	3.928	9.45	7.137	7.124	6.94	0.55
7	12.07	12.00	9.75	13.06	13.08	6.08	0.92	41.66	41.61	10.7	49.62	49.85	6.98	0.84
8	131.2	131.3	10.9	80.22	80.53	6.15	1.6	493.2	498.6	11.9	348.0	346.9	6.99	1.4
9	1579	1563	12.0	511.6	507.8	6.34	3.1	6316	6296	12.7	2466	2470	7.10	2.6
10	20900	20480	13.2	3640	3629	7.13	5.7	88320	88810	14.0	21590	21620	8.75	4.1

Figure 7: Computation times τ_n versus the number n of external gluons for the three different gluon color assignments used in Tables 5 (hard), 7 (simple) and 8 (random). The results reported in these tables are shown for the 4- and 5-dimensional color-ordered and -dressed algorithms. The solid and dashed curves each represent the outcomes of the fits listed in Table 9 for both the dressed and ordered approach, respectively.



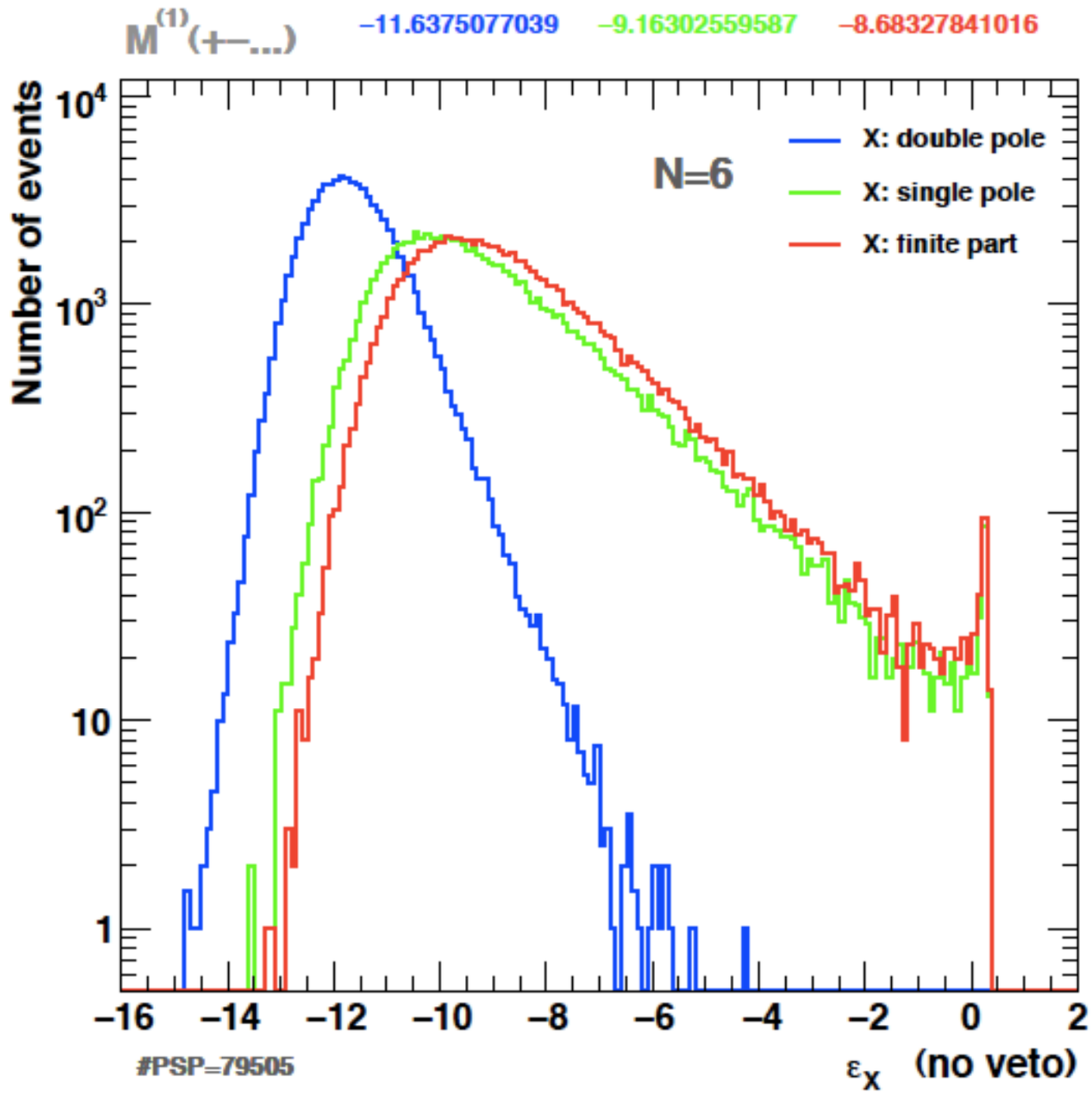
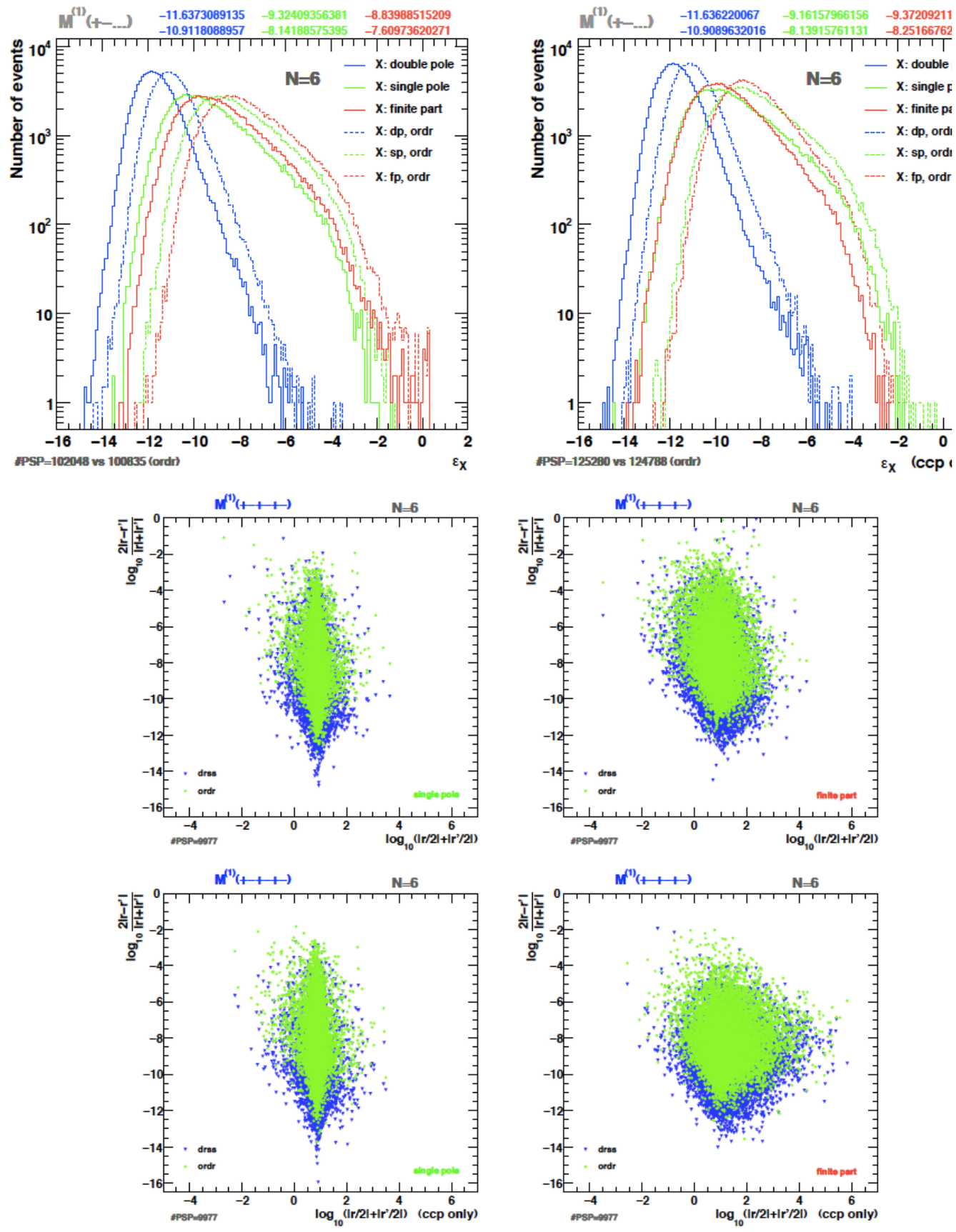
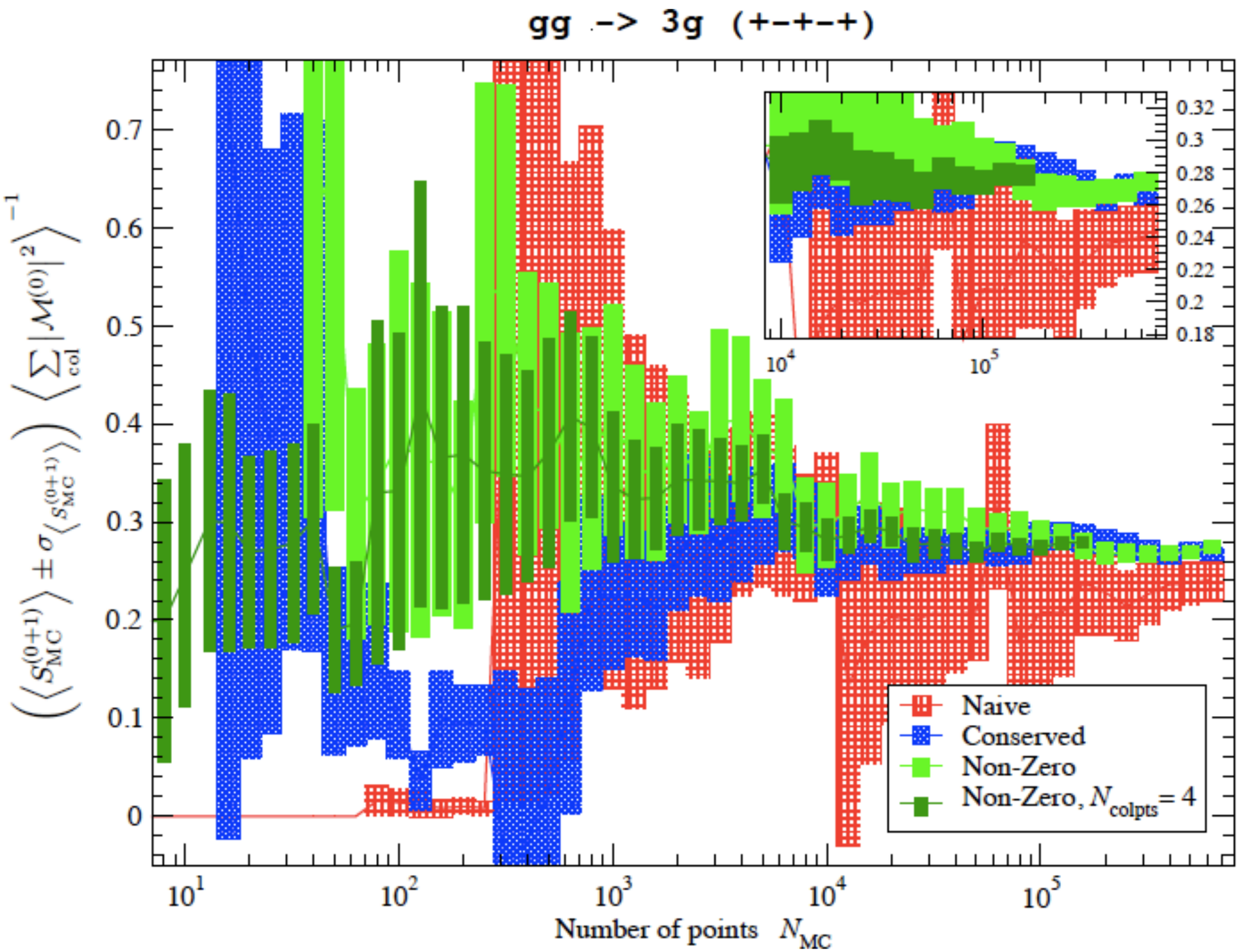
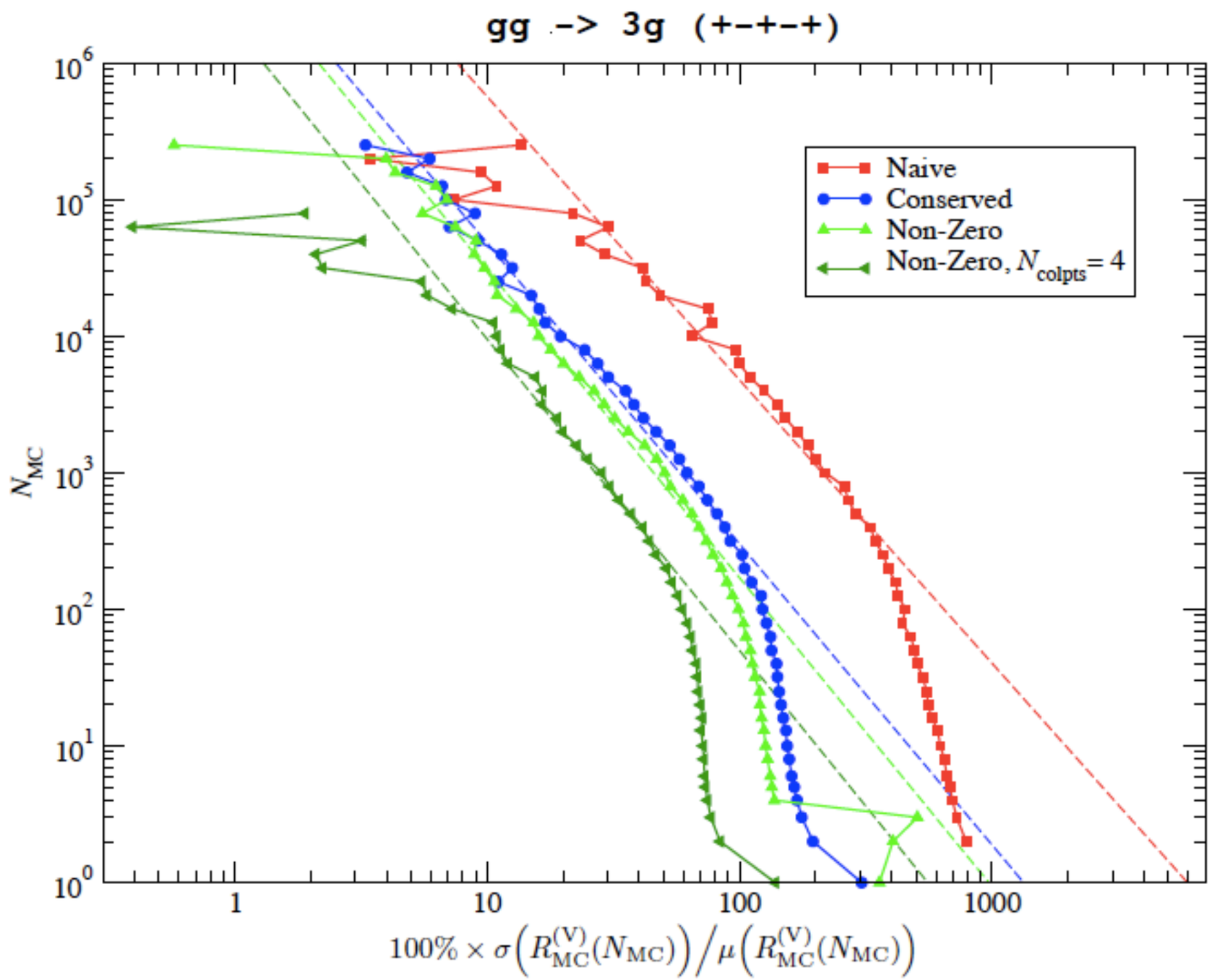


Figure 8: Relative accuracies of the $1/\epsilon^{2,1,0}$ poles of $n = 6$ gluon one-loop amplitudes as determined by the double-precision color-dressed algorithm. The gluon polarizations are given by $\lambda_k = + - + - + -$, colors were chosen randomly among non-zero configurations. Vetoed events are included, only those with unstable ortho-vectors are left out, see text for more explanations. The mean accuracies and the number of randomly picked phase-space points are displayed in the top row and bottom left corner of the plot, respectively.


M(1)(+−...)
−11.6373089135 −9.32409356381 −8.83988515209
−10.9118088957 −8.14188575395 −7.60973620271
N=6
X: double pole
X: single pole
X: finite part
X: dp, ordr
X: sp, ordr
X: fp, ordr
Number of events
εX
#PSP=102048 vs 100835 (ordr)
M(1)(+−...)
−11.636220067 −9.16157966156 −9.37209211
−10.9089632016 −8.13915761131 −8.25166762
N=6
εX (ccp
#PSP=125280 vs 124788 (ordr)
M(1)(+−+−)
N=6
drss
ordr
single pole
finite part
log10(|r/2|+|r'/2|)
#PSP=9977
log10(|r/2|+|r'/2|) (ccp only)


gg -> 3g (+-+-+)
Naive
Conserved
Non-Zero
Non-Zero, N_{colpts} = 4
Number of points N_{MC}
$\left(\left\langle S_{MC}^{(0+1)}\right\rangle \pm \sigma_{\left\langle S_{MC}^{(0+1)}\right\rangle}\right)\left\langle \sum_{col} |\mathcal{M}^{(0)}|^2 \right\rangle^{-1}$


gg -> 3g (+-+-+)
Naive
Conserved
Non-Zero
Non-Zero, N_{colpts}= 4
N_{MC}
10^0
10^1
10^2
10^3
10^4
10^5
10^6
1
10
100
1000
$100\% \times \sigma\left(R^{(V)}_{MC}(N_{MC})\right) / \mu\left(R^{(V)}_{MC}(N_{MC})\right)$

Concluding Remarks

- ☑ We have formulated D-dimensional unitarity to include color dressing

- ☑ The computer time scale like $\approx 7^n$ with the number of external particles The ordered formulation gives $\approx 9^n$ for n-gluon amplitudes. This is contrary to naive expectations since in the color dressed case we have larger number of cuts and internal color summation.

- ☑ Color-dressed formulation of DD unitarity is competitive for calculating one-loop virtual corrections for n-gluon scattering. It is expected it be even more efficient in processes involving less colorful particles.

- ☑ Color-dressed formulation has the same algorithm for particles of different color. The notion of primitive amplitudes is not needed.

- ☑ Color-dressed formulation appears to be better suited to full automation.