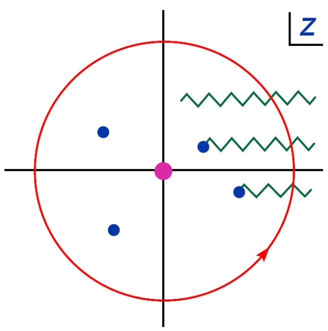
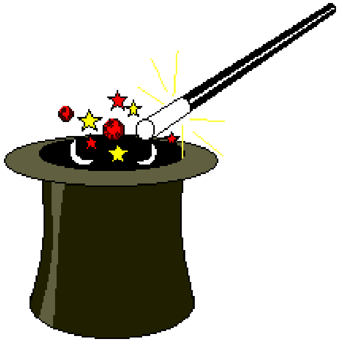
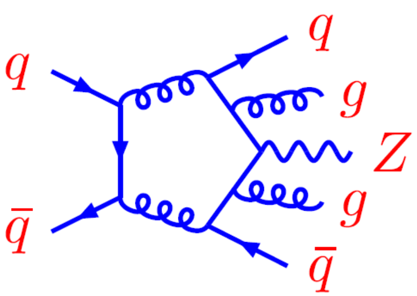
Unitarity and Recursive-Based Methods for Computing One-Loop Amplitudes





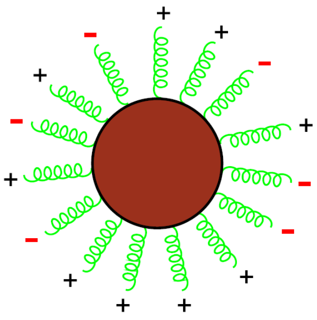


Lance Dixon (SLAC)
based on [**BlackHat**]:
C. Berger, Z. Bern, L.D., F. Febres Cordero, D. Forde,
H. Ita, D. Kosower, D. Maître, 0803.4180, 0808.0941

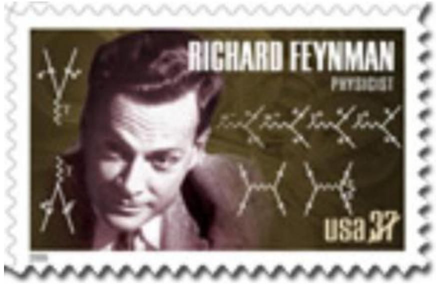
IPMU Focus Week on Jet Physics
November 12, 2009

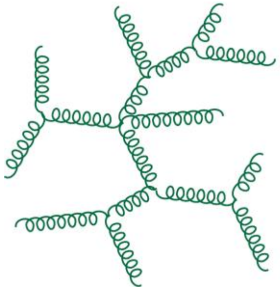
A better way to compute?

- Backgrounds (and many signals) require detailed understanding of scattering amplitudes for many ultra-relativistic ("massless") particles – especially quarks and gluons of QCD



- Feynman told us how to do this – in principle



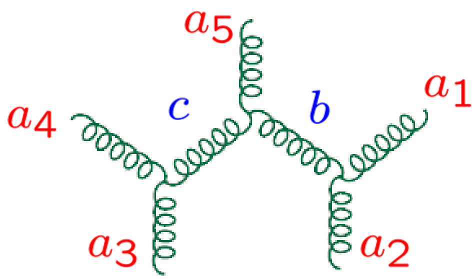


+ …

- Feynman rules, while very general and **wonderful**, are not optimized for these processes
- Can find more efficient methods, making use of analyticity + hidden symmetries (N=4 SUSY, twistor structure) of QCD

Color & Primitive Amplitudes

Standard color factor for a QCD graph has lots of structure constants contracted in various orders; for example:



$$\propto \; f^{a_1 a_2 b} f^{a_3 a_4 c} f^{b c a_5}$$

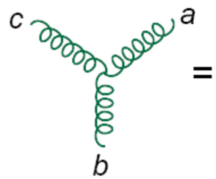
We can write every n-gluon tree graph color factor as a sum of traces of matrices T^a in the fundamental (defining) representation of $SU(N_c)$:

$\mathrm{Tr}(T^{a_1} T^{a_2} \cdots T^{a_n})$ + all non-cyclic permutations

Use definition: $[T^a, T^b] = i\, f^{abc}\, T^c$

+ normalization: $\mathrm{Tr}(T^a T^b) = \delta^{ab}$ → $\boxed{f^{abc} = -i\, \mathrm{Tr}([T^a, T^b]\, T^c)}$

Color in pictures

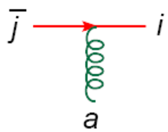
Insert 

$= f^{abc} = Tr([T^a, T^b]\ T^c) =$  - 

and

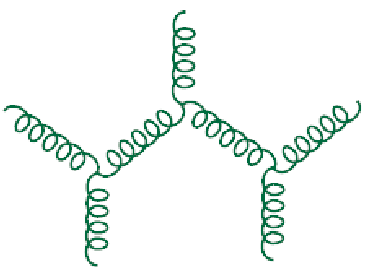
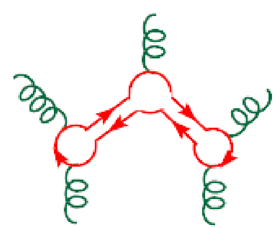
 = 

$-\frac{1}{N_c}$ 

where 

$= (T^a)_i^{\bar{j}}$ is color factor for *qqg* vertex

into typical string of f^{abc} structure constants for a Feynman diagram:

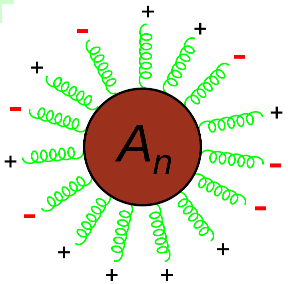
 =  + *permutations* $= \mathrm{Tr}(T^{a_1} T^{a_2} \ldots T^{a_n})$ + permutations

- Always single traces (at tree level)
- $\mathrm{Tr}(T^{a_1} T^{a_2} \ldots T^{a_n})$ comes only from those planar diagrams with cyclic ordering of external legs fixed to 1,2,…,n

Trace-based (dual) color decomposition

Similarly $\quad q\bar{q}gg\cdots g$ amplitudes $\Rightarrow (T^{a_1}T^{a_2}\cdots T^{a_n})_i^{\bar{j}}$ + permutations

In summary, for the *n*-gluon trees, the color decomposition is



$$\boxed{\mathcal{A}_n^{\text{tree}}(\{k_i, a_i, h_i\}) \;=\; g^{n-2}\,\text{Tr}(T^{a_1}T^{a_2}\cdots T^{a_n})\, A_n^{\text{tree}}(1^{h_1}, 2^{h_2}, \ldots, n^{h_n}) \;+\; \text{non-cyclic perm's}}$$

momenta

color

helicities
$h_i = \pm 1$

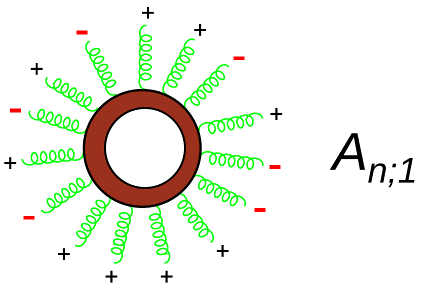
color-ordered primitive amplitude only depends on momenta.
Compute separately for each cyclicly inequivalent helicity configuration $\quad (h_1, h_2, \ldots, h_n)$

- Because $\quad A_n^{\text{tree}}(1^{h_1}, 2^{h_2}, \ldots, n^{h_n}) \quad$ comes from planar diagrams with cyclic ordering of external legs fixed to 1,2,…,n, it only has singularities in cyclicly-adjacent channels $s_{i,i+1}$, …

Trace-based color decomposition at one-loop

For *n*-gluon amplitudes at one loop, get a leading-color single-trace structure, plus subleading-color double trace structures:

$$\begin{aligned}\mathcal{A}_n^{1-\text{loop}}(\{k_i, a_i, h_i\}) \;=\;& g^n \{ N_c \mathrm{Tr}(T^{a_1} T^{a_2} \cdots T^{a_n})\, A_{n;1}(1^{h_1}, 2^{h_2}, \ldots, n^{h_n}) \\ &+ \text{non-cyclic perm's} \\ &+ \mathrm{Tr}(T^{a_1} T^{a_2})\, \mathrm{Tr}(T^{a_3} \ldots T^{a_n}) A_{n;3}(1^{h_1}, 2^{h_2}; 3^{h_3}, \ldots, n^{h_n}) \\ &+ \text{perm's} \\ &+ \cdots \}\end{aligned}$$



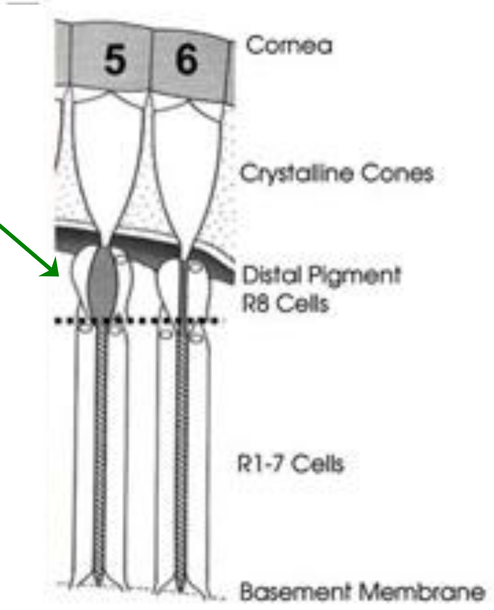
- $A_{n;1}(1^{h_1}, 2^{h_2}, \ldots, n^{h_n})$ also comes from planar 1-loop diagrams with cyclic ordering of external legs fixed to 1,2,…,n. So it also only has singularities in cyclicly-adjacent channels $s_{i,i+1}$, …

The Tail of the Mantis Shrimp

- Reflects left and right circularly polarized light differently

- Led biologists to discover that its eyes have differential sensitivity
- It communicates via the helicity formalism

$\lambda/4$ plate



"It's the most private communication system imaginable. No other animal can see it."

- Roy Caldwell (U.C. Berkeley)

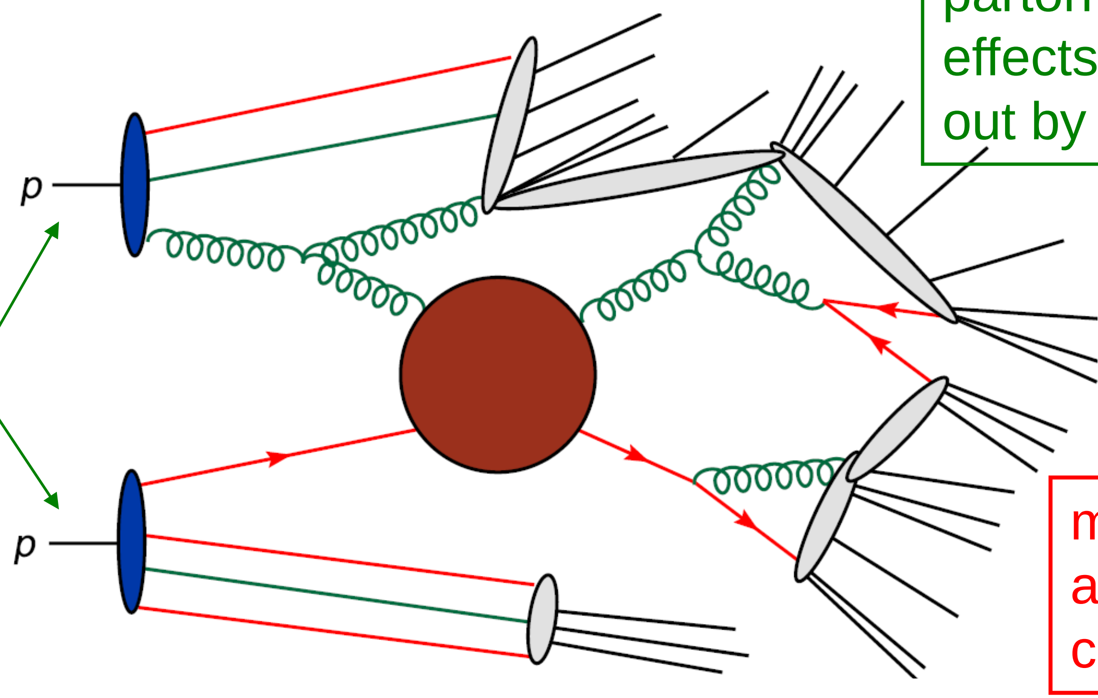
What the Biologists Didn't Know

Particle theorists have also evolved capability to communicate results via helicity formalism

LHC experimentalists are blind to it

any final-state parton polarization effects washed out by fragmentation

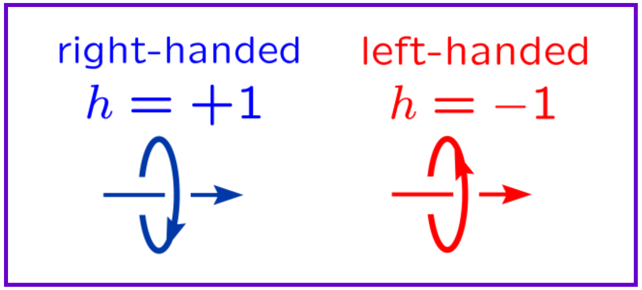
unpolarized

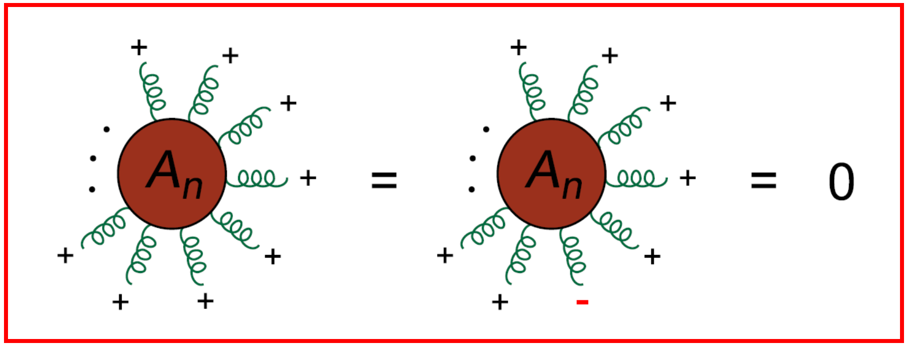


must sum over all helicity configurations

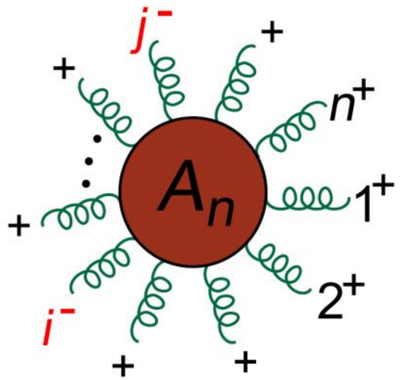
Helicity Formalism
→ Tree-Level Simplicity in QCD

Many helicity amplitudes either vanish or are very short





Analyticity makes it possible to recycle this simplicity into loop amplitudes

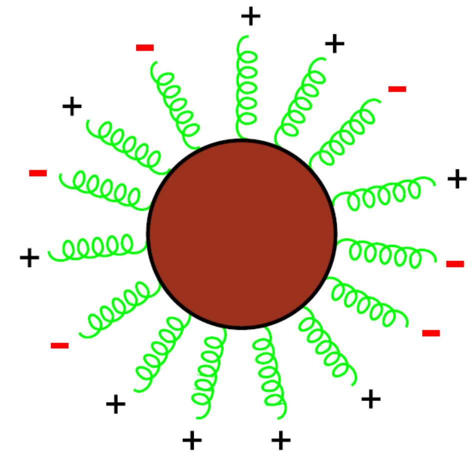


$$= \frac{\langle i\, j \rangle^4}{\langle 1\, 2 \rangle \langle 2\, 3 \rangle \cdots \langle n\, 1 \rangle}$$

Parke-Taylor formula (1986)

The right variables

Scattering amplitudes for massless
plane waves of definite 4-momentum:
Lorentz vectors k_i^μ $\qquad k_i^2 = 0$



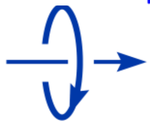
Textbook: use Lorentz-invariant products
(invariant masses): $s_{ij} = 2k_i \cdot k_j = (k_i + k_j)^2$

But for particles with **spin**
there are better variables

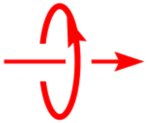
massless q,g,γ
all have 2 helicities

Take "square root" of 4-vectors k_i^μ (spin 1)
use 2-component Dirac (Weyl) spinors $u_\alpha(k_i)$ (spin ½)

right-handed: $(\lambda_i)_\alpha = u_+(k_i)$
$h = +1/2$



left-handed: $(\tilde{\lambda}_i)_{\dot{\alpha}} = u_-(k_i)$
$h = -1/2$



Spinor products

Instead of Lorentz products:

$$s_{ij} = 2k_i \cdot k_j = (k_i + k_j)^2$$

Use spinor products:

$$\boxed{\begin{aligned} \varepsilon^{\alpha\beta}(\lambda_i)_\alpha(\lambda_j)_\beta &= \langle i\, j\rangle \\ \varepsilon^{\dot\alpha\dot\beta}(\tilde\lambda_i)_{\dot\alpha}(\tilde\lambda_j)_{\dot\beta} &= [i\, j] \end{aligned}}$$

These are **complex square roots** of Lorentz products (if k_i real):

$$\langle i\, j\rangle = \sqrt{s_{ij}}\, e^{i\phi_{ij}} \qquad [j\, i] = \sqrt{s_{ij}}\, e^{-i\phi_{ij}}$$

Analyticity

- **If we think of λ_i and $\tilde{\lambda}_i$ as independent variables, then momenta k_i^μ must be thought of as complex**

(for real momenta, λ_i and $\tilde{\lambda}_i$ are complex conjugates)

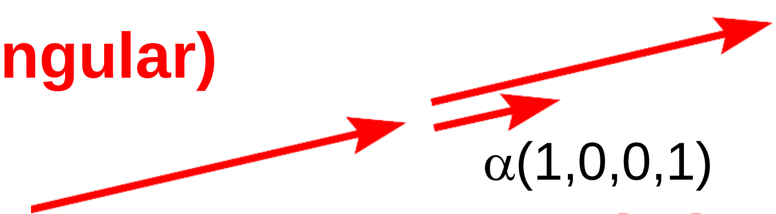
- **There are special complex kinematics, dictated by how λ_i and $\tilde{\lambda}_i$ behave**

Virtues of complex momenta

- Makes sense of most basic process with all 3 particles massless

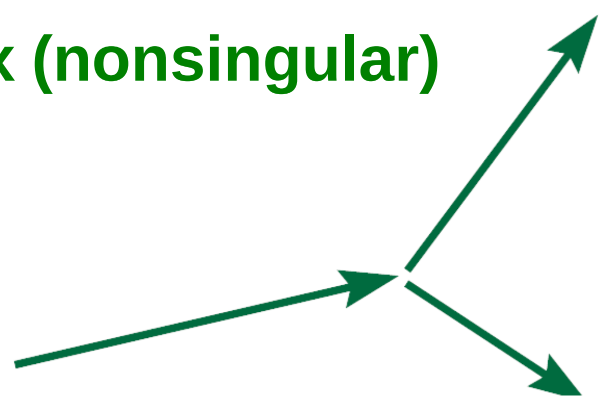
$$s_{ij} = 2k_i \cdot k_j = (k_i + k_j)^2 = 0 \quad \forall\ i,j$$

real (singular)



$$\langle i\, j\rangle = [i\, j] = s_{ij} = 0 \qquad \forall\ i,j$$

complex (nonsingular)



$$\tilde\lambda_1 \propto \tilde\lambda_2 \propto \tilde\lambda_3$$

$$[i\, j] = 0 \quad \text{but} \quad \langle i\, j\rangle \neq 0$$

$$\boxed{\frac{\langle i\, j\rangle^4}{\langle 1\,2\rangle\langle 2\,3\rangle\langle 3\,1\rangle} \text{ makes sense}}$$

For Efficient Computation

Reduce

the number of “diagrams”

Reuse

building blocks over & over

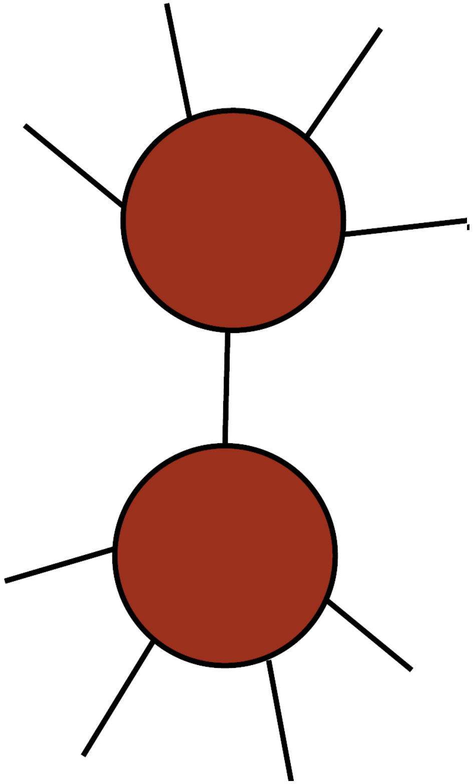
Recycle

lower-point (1-loop) & lower-loop (tree)
on-shell amplitudes

Recurse

Factorization

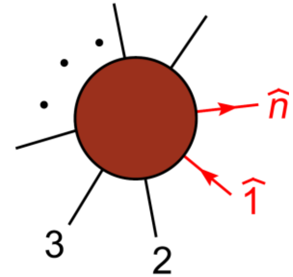
Amplitudes are "plastic".
Fall apart into simpler ones in special limits.



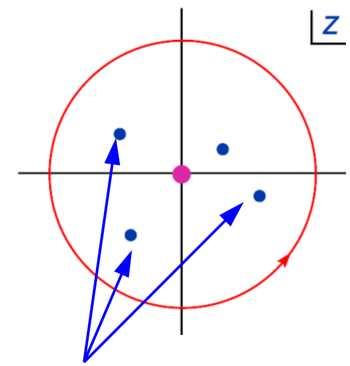
Explore limits in complex plane

Britto, Cachazo, Feng, Witten, hep-th/0501052

Inject complex momentum at leg 1, remove it at leg *n.*



$$k_1(z) + k_n(z) \; = \; k_1 + k_n \qquad \Rightarrow \qquad A(0) \;\rightarrow\; A(z)$$
$$k_1^2(z) = k_n^2(z) = 0$$

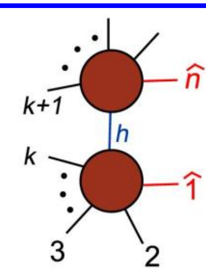


special limits $\Leftrightarrow$ poles in z

Cauchy: If $A(\infty) = 0$ then

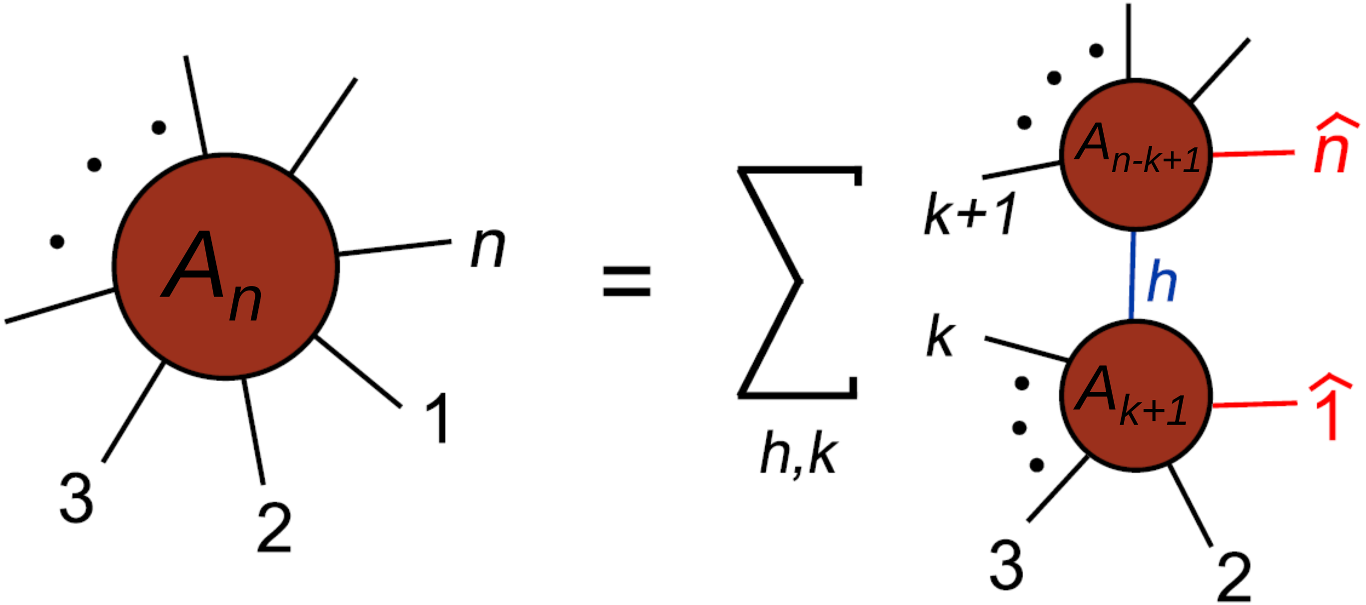
$$0 = \frac{1}{2\pi i} \oint dz \, \frac{A(z)}{z} \;=\; A(0) + \sum_k \mathrm{Res} \left[\frac{A(z)}{z} \right] \Big|_{z=z_k}$$

$\boxed{\textbf{residue at } z_k \;=\; [k^{\mathrm{th}} \text{ factorization limit}] \;=}$



→ BCFW (on-shell) recursion relations

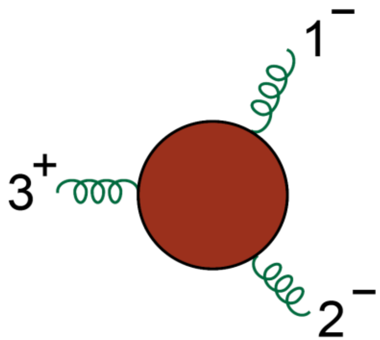
Britto, Cachazo, Feng, hep-th/0412308



A_{k+1} and A_{n-k+1} are **on-shell** tree amplitudes with **fewer** legs, and with momenta shifted by a **complex** amount

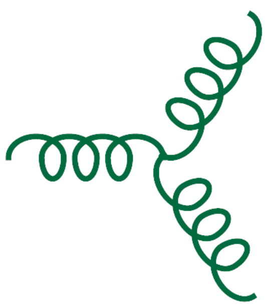
Trees recycled into trees

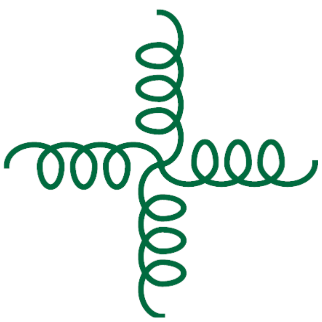
All gluon tree amplitudes built from:



$$= \frac{\langle 1\,2\rangle^4}{\langle 1\,2\rangle\langle 2\,3\rangle\langle 3\,1\rangle}$$

(In contrast to Feynman vertices, it is on-shell, gauge invariant.)



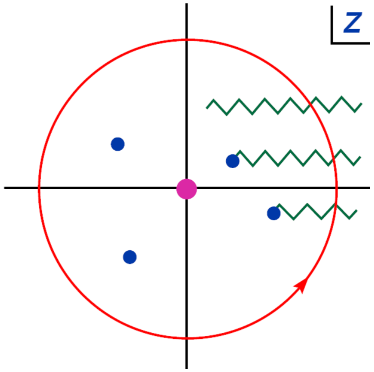


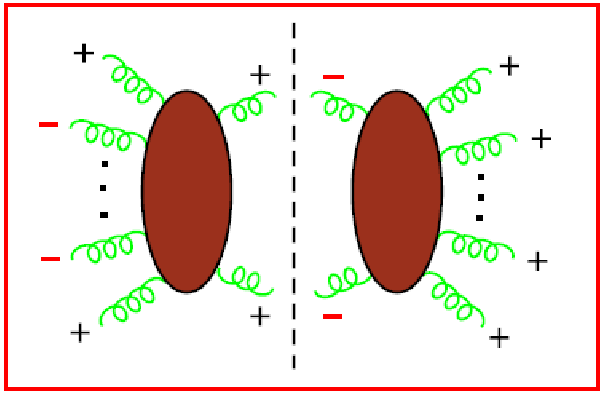
On-shell recursion at one loop

Bern, LD, Kosower, hep-th/0501240, hep-ph/0505055, hep-ph/0507005;
Berger, et al., hep-ph/0604195, hep-ph/0607014

- **Same techniques** work for one-loop QCD amplitudes

- **New features** compared with tree case, especially **branch cuts**
- Determine cut terms efficiently using (generalized) unitarity





Trees recycled into loops!

One-loop amplitude decomposition

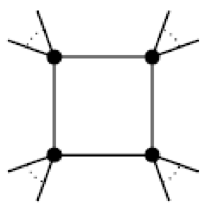
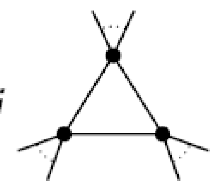
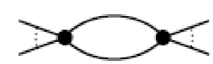
When all external momenta are in *D=4*, loop momenta in *D=4-2ε*
(dimensional regularization), one can write: BDDK (1994)

rational part

cut part

known **scalar** one-loop integrals,
same for all amplitudes

$$A = R + C + \mathcal{O}(\epsilon)$$

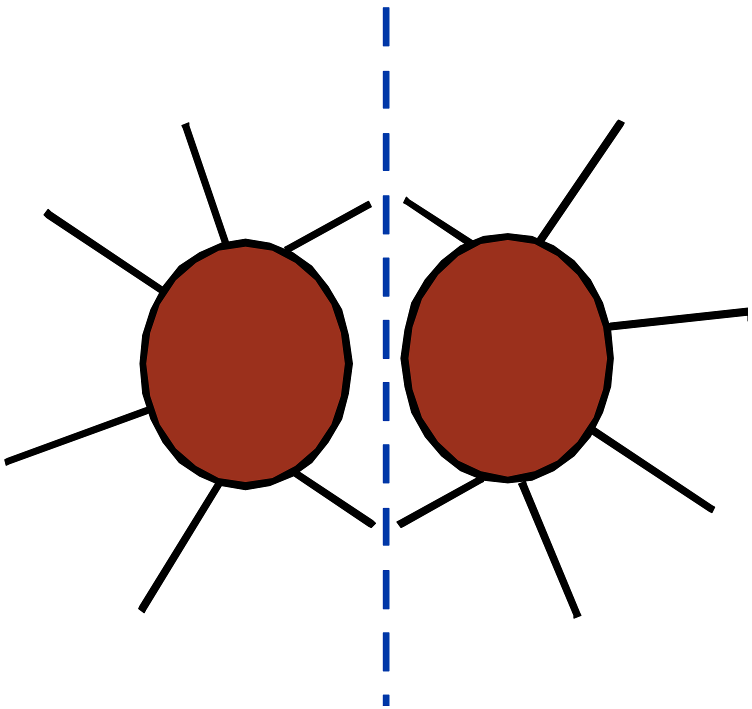
$$C = \sum_i d_i$$  $$+ \sum_i c_i$$  $$+ \sum_i b_i$$ 

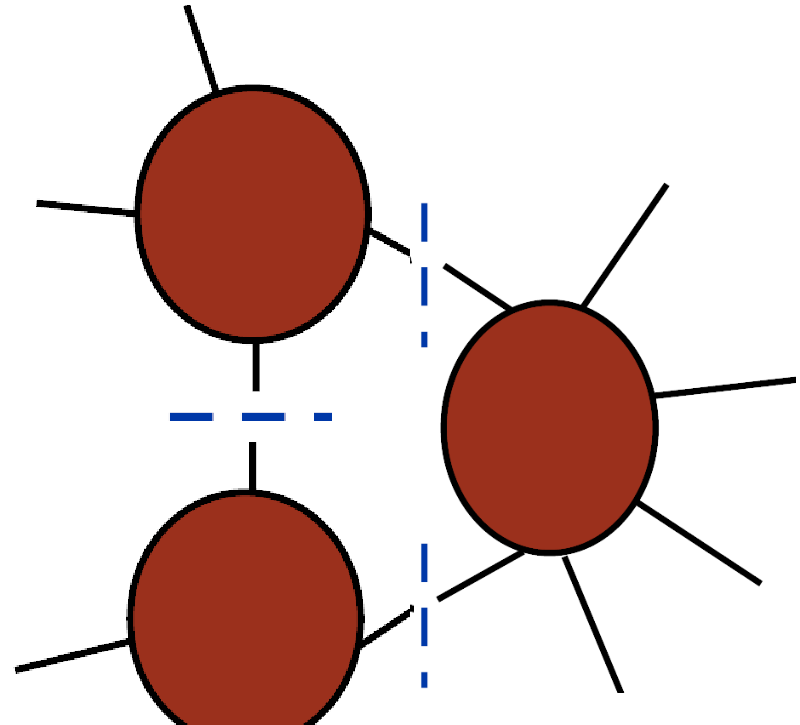
determine coefficients – all rational functions
– using (generalized) unitarity

Generalized unitarity at one loop

Ordinary unitarity:
put 2 particles on shell

Generalized unitarity:
put 3 or 4 particles on shell



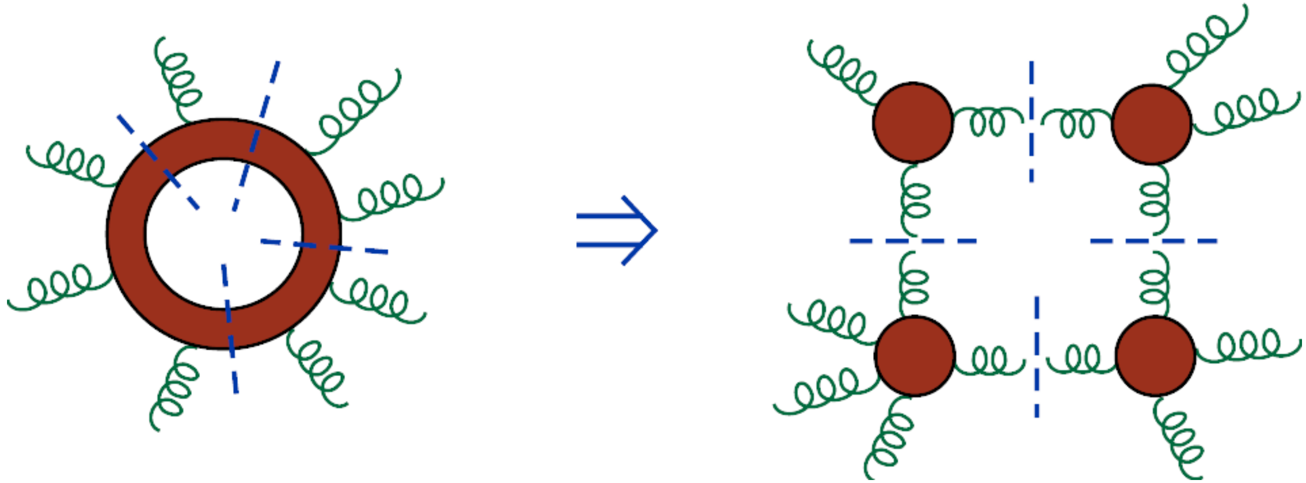


trees get simpler

cut conditions require complex loop momenta

Generalized unitarity for box coefficients b_i

Britto, Cachazo, Feng, hep-th/0412308

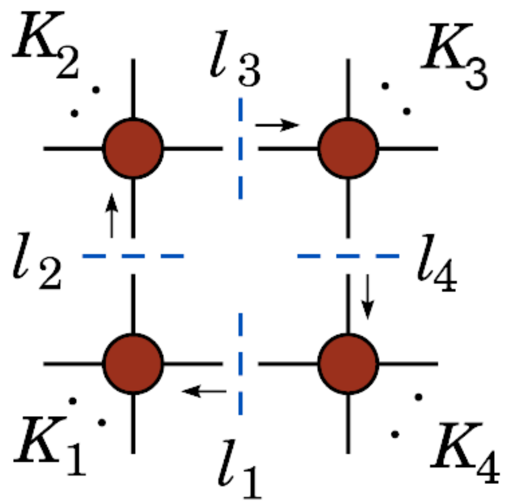


$$
\begin{aligned}
&\int d^4\ell\ \delta(\ell_1^2 - m_1^2)\,\delta(\ell_2^2 - m_2^2) \\
&\qquad \times\delta(\ell_3^2 - m_3^2)\,\delta(\ell_4^2 - m_4^2)\ \times A^{1-\text{loop}}(\ell_i) \\
= \quad & A_1^{\text{tree}}(\ell_0)A_2^{\text{tree}}(\ell_0)A_3^{\text{tree}}(\ell_0)A_4^{\text{tree}}(\ell_0) \\
= \quad & d_i
\end{aligned}
$$

no. of dimensions = 4 = no. of constraints → discrete solutions (2)

Box coefficients b_i (cont.)

For improved numerical stability, can use simplified solutions when all internal lines massless, at least one external line (K_1) massless:



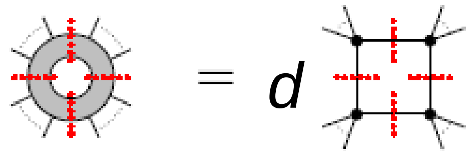
$$(l_1^{(\pm)})^\mu = \frac{\langle 1^\mp | \not{K}_2 \not{K}_3 \not{K}_4 \gamma^\mu | 1^\pm \rangle}{2 \langle 1^\mp | \not{K}_2 \not{K}_4 | 1^\pm \rangle}, \qquad (l_2^{(\pm)})^\mu = -\frac{\langle 1^\mp | \gamma^\mu \not{K}_2 \not{K}_3 \not{K}_4 | 1^\pm \rangle}{2 \langle 1^\mp | \not{K}_2 \not{K}_4 | 1^\pm \rangle},$$

$$(l_3^{(\pm)})^\mu = \frac{\langle 1^\mp | \not{K}_2 \gamma^\mu \not{K}_3 \not{K}_4 | 1^\pm \rangle}{2 \langle 1^\mp | \not{K}_2 \not{K}_4 | 1^\pm \rangle}, \qquad (l_4^{(\pm)})^\mu = -\frac{\langle 1^\mp | \not{K}_2 \not{K}_3 \gamma^\mu \not{K}_4 | 1^\pm \rangle}{2 \langle 1^\mp | \not{K}_2 \not{K}_4 | 1^\pm \rangle}.$$

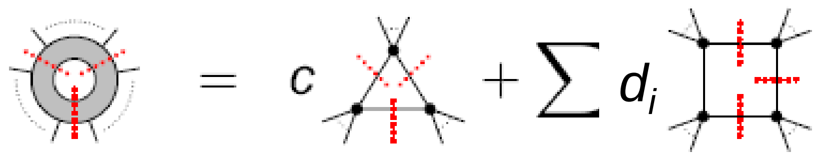
BH, 0803.4180; Risager 0804.3310

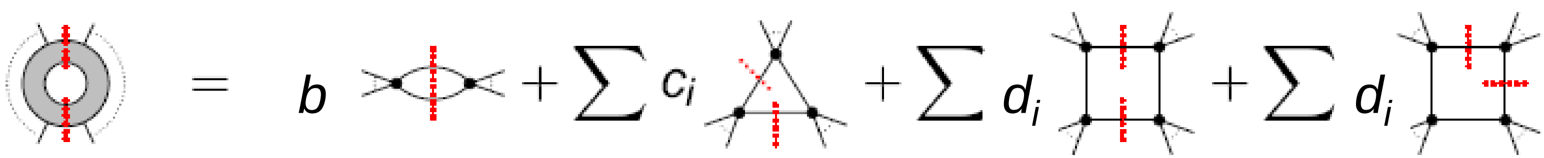
Triangles & bubbles

With a 4-ple cut we select one coefficient



Triangle and bubble coefficients are more complicated since a double or triple cut does not isolate a single coefficient.





Also, solutions to cut constraints are now continuous,
so there are multiple ways to solve and eliminate d_i , etc.

Britto et al. (2005,2006); Ossola, Papadopoulos, Pittau, hep-ph/0609007;
Mastrolia hep-th/0611091;
Forde, 0704.1835; Ellis, Giele, Kunszt, 0708.2398; …

Triangle coefficients

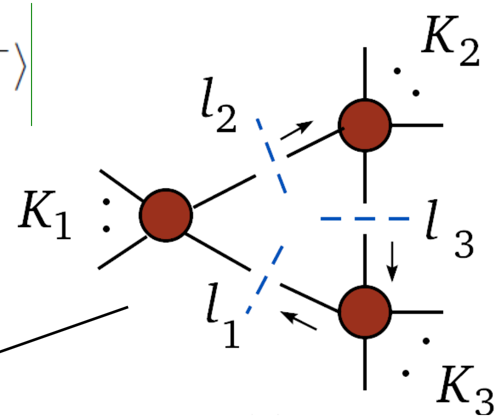
Forde, 0704.1835; BH, 0803.4180

Triple cut solution depends on one complex parameter, t

$$l_1^\mu(t) \;=\; \tilde{K}_1^\mu + \tilde{K}_3^\mu + \frac{t}{2}\langle \tilde{K}_1^- | \gamma^\mu | \tilde{K}_3^- \rangle + \frac{1}{2t}\langle \tilde{K}_3^- | \gamma^\mu | \tilde{K}_1^- \rangle$$

Solves $\quad l_1^2(t) \;=\; l_2^2(t) \;=\; l_3^2(t) \;=\; 0$

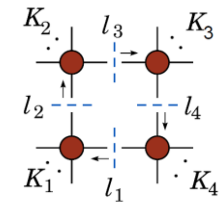
for suitable definitions of $\quad \tilde{K}_1^\mu \;\; \tilde{K}_3^\mu$



$$C_3(t) \equiv A_{(1)}^{\rm tree} A_{(2)}^{\rm tree} A_{(3)}^{\rm tree} \Big|_{l_i = l_i(t)}$$

Box-subtracted triple cut has poles only at $t = 0, \infty$

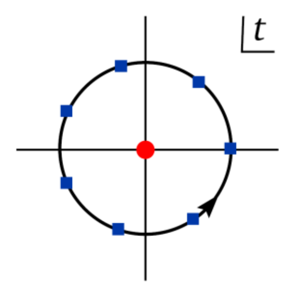
$$\boxed{T_3(t) \equiv C_3(t) - \sum_{\sigma=\pm} \sum_i \frac{d_i^\sigma}{\xi_i^\sigma (t - t_i^\sigma)}}$$



$$\boxed{T_3(t) = \sum_{j=-p}^{p} c_j t^j}$$

Bubble similar

Triangle coefficient c_0
plus all other coefficients c_j
obtained by discrete Fourier projection, sampling at
$(2p+1)^{\rm th}$ roots of unity



Rational function *R*

No cuts in D=4 – can't get from D=4 unitarity

However, can get using D=4-2ε unitarity:

$$\int d^{4-2\epsilon}\ell \quad \Rightarrow \quad R(s_{ij}) \;\to\; R(s_{ij})\,(-s_{12})^{-\epsilon} \;=\; R(s_{ij})\,[1-\epsilon\ln(-s_{ij})]$$

Bern, Morgan (1996); Bern, LD, Kosower (1996);
Brandhuber, McNamara, Spence, Travaglini hep-th/0506068;
Anastasiou, Britto, Feng, Kunszt, Mastrolia, hep-th/0609191, hep-th/0612277;
Britto, Feng, hep-ph/0612089, 0711.4284;
Giele, Kunszt, Melnikov, 0801.2237;
Britto, Feng, Mastrolia, 0803.1989; Britto, Feng, Yang, 0803.3147;
Giele, Kunszt, Melnikov (2008); Giele, Zanderighi, 0805.2152;
Ellis, Giele, Kunszt, Melnikov, 0806.3467;
Feng, Yang, 0806.4106; Badger, 0806.4600;
Ellis, Giele, Kunszt, Melnikov, Zanderighi, 0810.2762;
Kunszt talk at this workshop; Kunszt review (to appear)

OR: Get rational function R using on-shell recursion

• Used to get infinite series of one-loop QCD helicity amplitudes analytically:

- n-gluon MHV amplitudes at 1-loop $(- + \cdots + - + \cdots +)$
- n-gluon "split" helicity amplitudes $(- - \cdots - + + \cdots +)$
- "Higgs" + n-gluon MHV amplitudes $(\phi; - + \cdots + - + \cdots +)$

Forde, Kosower, hep-ph/0509358;
Berger, Bern, LD, Forde, Kosower, hep-ph/0604195, hep-ph/0607014;
Badger, Glover, Risager, 0704.3194;
Glover, Mastrolia, Williams, 0804.4149

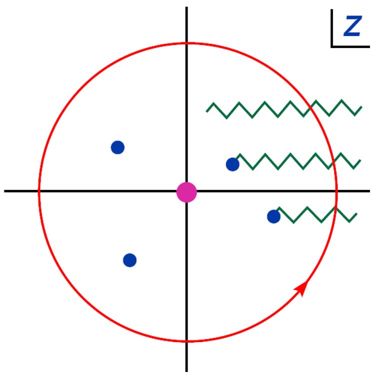
• Also other specific amplitudes analytically
• Method can be implemented numerically as well (BlackHat)

Loop amplitudes with cuts

Generic analytic properties of shifted 1-loop amplitude, $A_n(z)$

Cuts and poles in z-plane:

$$\ln(s_{23}) \Rightarrow \ln[(\langle 2\,3\rangle + z\langle 1\,3\rangle)[3\,2]]$$



But if we know the cuts (*via* unitarity in D=4), we can subtract them: $R_n \equiv A_n - C_n$

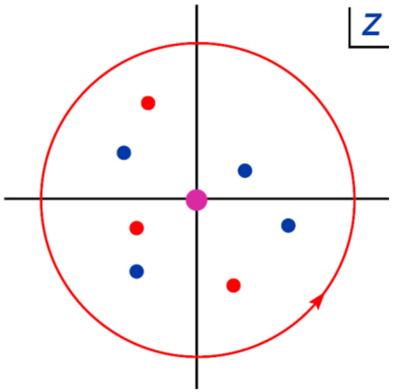
rational part

full amplitude

cut-containing part

Shifted rational function $R_n(z) = A_n(z) - C_n(z)$

has no cuts, but has spurious poles in z because of C_n:

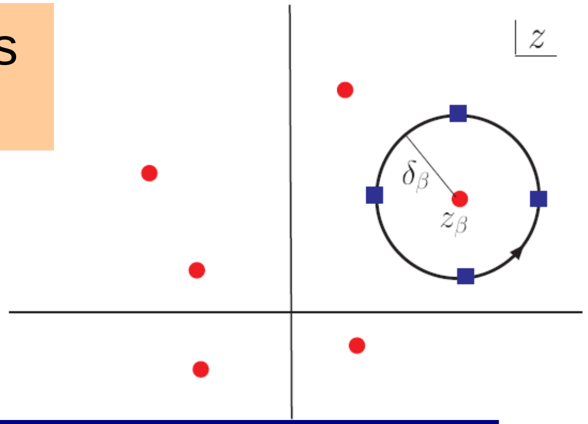


$$C_n \longrightarrow \frac{\ln(r) + 1 - r}{(1-r)^2} \longleftarrow R_n$$

Gram det

Spurious poles

Locations all known (from Gram determinants associated with various scalar integrals)



Residues determined using cut part (residues cancel in full amplitude):

$$R_n^S(0) = - \sum_{\text{spur. poles } \beta} \operatorname*{Res}_{z=z_\beta} \frac{R_n(z)}{z} = \sum_{\text{spur. poles } \beta} \operatorname*{Res}_{z=z_\beta} \frac{C_n(z)}{z}$$

Loop integrals appear in C_n. We expand them around the spurious poles, keeping only rational parts. E.g. for 3-mass triangle integral:

$$I_3^{3\mathrm{m}}(s_1,s_2,s_3) \to -\frac{1}{2}\sum_{i=1}^{3}\ln(-s_i)\frac{s_i - s_{i+1} - s_{i-1}}{s_{i+1}s_{i-1}}\left[1 - \frac{1}{6}\frac{\Delta_3}{s_{i+1}s_{i-1}} + \frac{1}{30}\left(\frac{\Delta_3}{s_{i+1}s_{i-1}}\right)^2\right]$$

as $\Delta_3 \to 0$

$$+\frac{1}{6}\frac{\Delta_3}{s_1s_2s_3} - \frac{s_1+s_2+s_3}{120}\left(\frac{\Delta_3}{s_1s_2s_3}\right)^2 + \cdots,$$

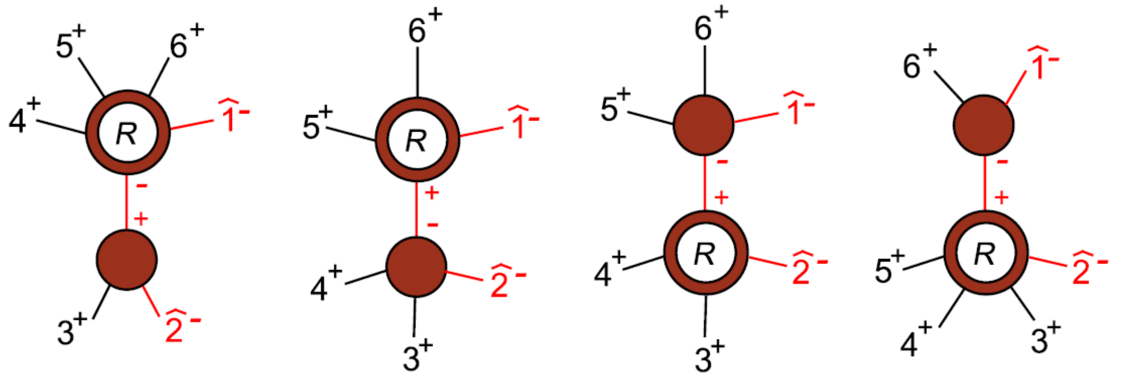
$$\boxed{\Delta_3 = \sum_{i=1}^{3}(s_i^2 - 2s_is_{i+1})}$$

Physical poles → recursive diagrams

Example

For rational part of $A_6^{1-\mathrm{loop}}(1^-, 2^-, 3^+, 4^+, 5^+, 6^+)$

there are only four recursive diagrams:



loops recycled into loops

(Compared with 10,860 1-loop Feynman diagrams)

Several Related Implementations

CutTools: Ossola, Papadopolous, Pittau, 0711.3596

NLO *WWW* Binoth+OPP, 0804.0350

NLO *ttbb* Bevilacqua, Czakon, Papadopoulos, Pittau, Worek, 0907.4723

Rocket: Giele, Zanderighi, 0805.2152

One-loop n-gluon amplitudes for n up to 20;

W + 3 jets amplitudes

Ellis, Giele, Kunszt, Melnikov, Zanderighi, 0810.2762

NLO *W* + 3 jets in leading-color (large N_c) approximation

Ellis, Melnikov, Zanderighi, 0901.4101, 0906.1445

Melnikov, Zanderighi, 0910.3671

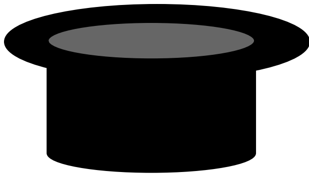
D-dim'l unitarity

Blackhat: Berger, Bern, LD, Febres Cordero, Forde, H. Ita, D. Kosower, D. Maître, 0803.4180, 0808.0941

One-loop n-gluon amplitudes for n up to 7,…;

amplitudes needed for NLO production of *W,Z* + 3 jets

D-dim'l unitarity + on-shell recursion

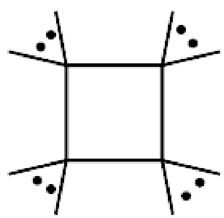
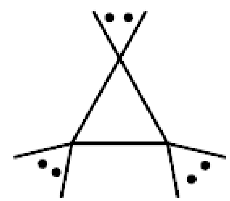
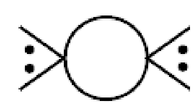


Practical issues

• **Evaluation time** (for Monte Carlo integration over phase space)

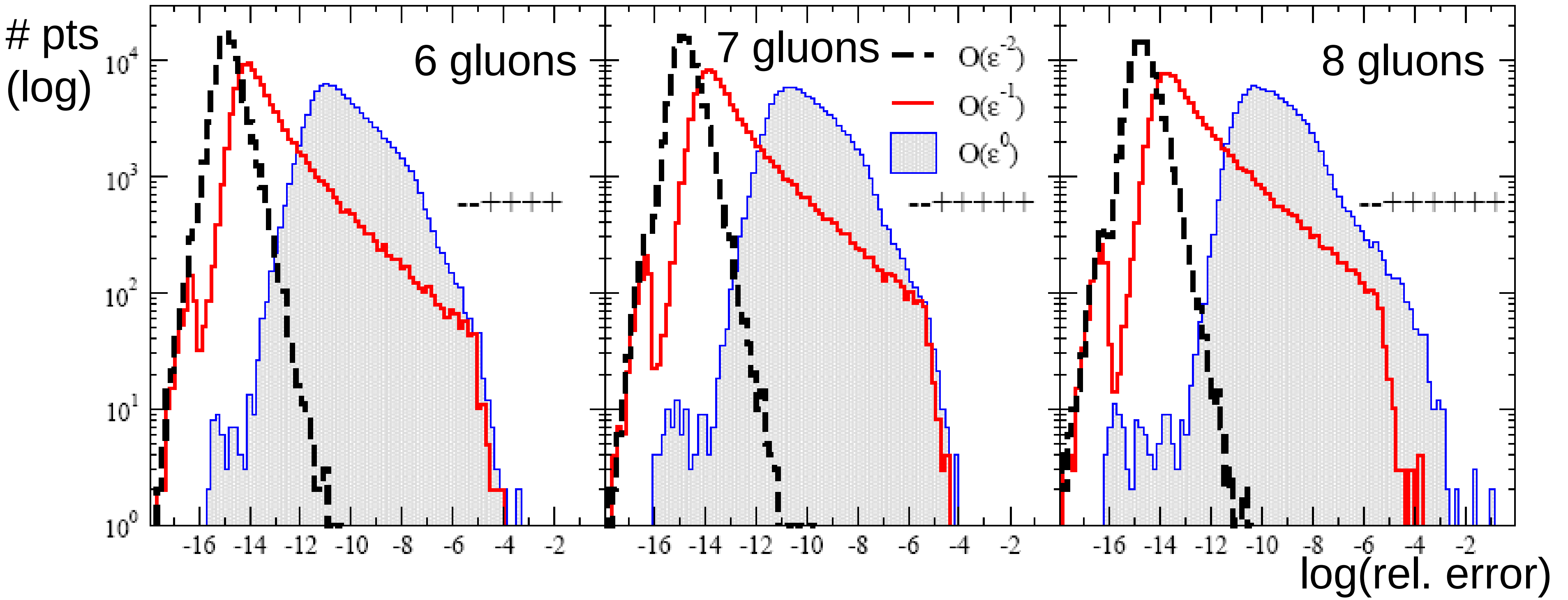
• **Numerical imprecision due to round-off errors**
– there can be **large cancellations** between different cut terms, and also against rational terms, in special phase space regions
– especially where the **Gram determinants** associated with the box and triangle integrals **vanish**:

$$\Delta = \det(p_i . p_j) \quad \to \quad 0$$

$$A^{1\text{-loop}} = \sum_i d_i$$  $$+ \sum_i c_i$$  $$+ \sum_i b_i$$ 

$$+ \; R \; + \mathcal{O}(\epsilon)$$

BlackHat results for n gluons



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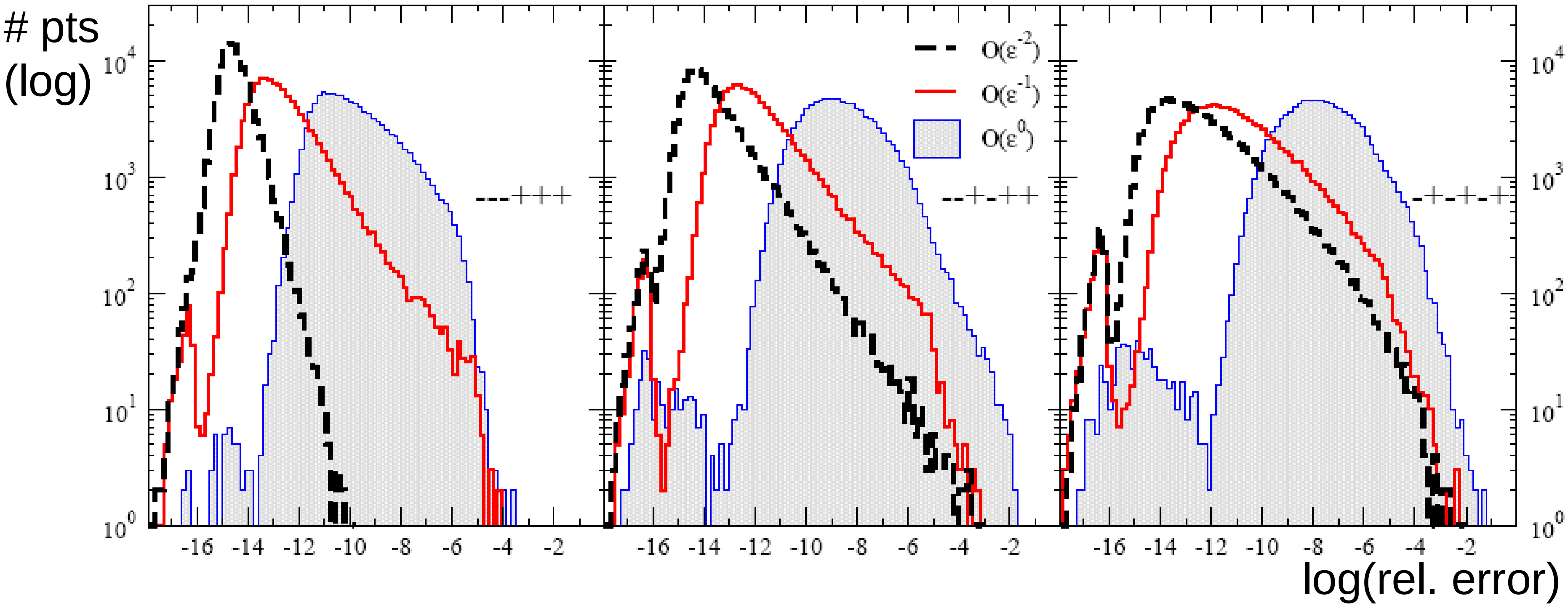
Most complex 6-gluon helicity amplitudes



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See also Giele, Zanderighi, 0805.2152

Conclusions

- New and efficient computational approaches to gauge theories based on analyticity – unitarity and factorization.
- Now, state-of-art one-loop amplitudes in QCD – needed for important collider applications – can be computed by these techniques, and have begun to be applied to full NLO results.
- Real radiation also required (but not discussed here)

– many automated programs for building a set of counterterms have been constructed recently:

Gleisberg, Krauss, 0709.2881; Seymour, Tevlin, 0803.2231;
Hasegawa, Moch, Uwer, 0807.3701;
Frederix, Gehrmann, Greiner, 0808.2128;
Czakon, Papadopoulos, Worek, 0905.0883;
Frederix, Frixione, Maltoni, Stelzer, 0908.4272 [MadFKS];
Frederix talk at this workshop

- **Still need to automate things further, in order to construct NLO programs for wider classes still of important LHC background processes.**