# On the structure of an elliptic fibration

§1 Smooth elliptic fibrations
§2 Weierstrass models
§3 local structure
§4 global structure

Ref. "On Weierstrass models" in Algebraic Geometry and Commutative Algebra in Honor of M. Nagata, Kinokuniya 1988 vol 2, 405–431.

"Local structure of an elliptic fibration" in Higher Dimensional Birational Geometry, Adv. Stud. in Pure Math 35, 2002, 185–295.

"Global structure of an elliptic fibration" Publ. RIMS Kyoto Univ. vol 38, No. 3 (2002) 451–649.

## §1 Smooth elliptic fibrations

### §1.1 Elliptic curves

- An elliptic curve = a compact Riemann surface of genus 1
  (nonsingular projective curve / $\mathbb{C}$)
- A polarized Hodge structure of weight 1, rank 2:

$$H = (H_{\mathbb{Z}}, Q, F)$$

$H_{\mathbb{Z}}$ : a free abelian group of rank 2 $(\cong \mathbb{Z}^{\oplus 2})$

$Q$ : (polarization) $H_{\mathbb{Z}} \times H_{\mathbb{Z}} \longrightarrow \mathbb{Z}$ $\quad (\wedge^2 H_{\mathbb{Z}} \simeq \mathbb{Z})$

nonsingular skew symmetric bilinear form

$F$ : (Hodge filtration) $F \subseteq H_{\mathbb{Z}} \otimes \mathbb{C}$ 1-dim. subspace

$$F^0 = H_{\mathbb{Z}} \otimes \mathbb{C} \supseteq F^1 = F \supseteq F^2 = 0$$

such that $\begin{cases} \cdot\ H_{\mathbb{Z}} \otimes \mathbb{C} = F \oplus \overline{F} \\ \cdot\ \sqrt{-1}\, Q(x, \bar{x}) > 0 \text{ for any } 0 \neq x \in F. \end{cases}$

"An elliptic curve defines a polarized Hodge structure of wt 1, rk 2":
(HS)

X : elliptic curve $\quad H_{\mathbb{Z}} = H^1(X, \mathbb{Z})$

$\Downarrow \quad Q : H^1(X,\mathbb{Z}) \times H^1(X,\mathbb{Z}) \xrightarrow{\cup} H^2(X,\mathbb{Z}) \xrightarrow{\sim} \mathbb{Z}$ natural orientation

$(H_{\mathbb{Z}}, Q, F)$ $\quad F^1 = H^0(X, \Omega^1_{X/\mathbb{C}}) \subseteq H^1(X, \mathbb{C})$

pol HS. $\quad\Uparrow$

$(0 \to \mathbb{C}_X \to \mathcal{O}_X \xrightarrow{d} \Omega^1_{X/\mathbb{C}} \to 0)$ Poincaré exact seq.

"A pol. HS wt 1 rk 2 defines an elliptic curve":

$H = (H_{\mathbb{Z}}, Q, F)$

$\Longrightarrow X = H_{\mathbb{C}} / (H_{\mathbb{Z}} + F)$ is a 1-dimensional complex torus = an elliptic curve.

Rem. $\{\text{elliptic curve}\}/\sim_{\text{isomorphism}} \underset{\substack{1-1 \\ \text{corresp.}}}{\longleftrightarrow} \{\text{pol HS wt 1 rk 2}\}/\sim_{\text{isom}}$

$$H^1(X, \mathbb{Z}) \hookrightarrow H^1(X, \mathcal{O}_X) \underset{\text{corresp}}{\longleftrightarrow} H_{\mathbb{Z}} \hookrightarrow H_{\mathbb{C}}/F$$

"An elliptic curve is a nonsingular plane cubic curve":

$X \ni p$ point = a prime divisor $\Rightarrow \mathcal{O}_X(3p)$ very ample

$\Rightarrow \exists\, \iota : X \hookrightarrow \mathbb{P}^2$ s.t. $\iota^* \mathcal{O}_{\mathbb{P}^2}(1) \cong \mathcal{O}_X(3p)$ $\Rightarrow$ X is cubic

"A nonsingular plane cubic curve is an elliptic curve."

## §1.2 Smooth basic elliptic fibrations

$S$: complex manifold ( = a connected Hausdorff non singular complex analytic space )

**Def** A smooth elliptic fibration $\pi : X \to S$ is a proper smooth morphism whose fibers are all elliptic curves

It is called basic if there is a section ($\sigma : S \to X$ s.t. $\pi \circ \sigma = id_S$)

- A polarized variation of Hodge structure (VHS) of wt 1, rk2 on $S$:

$$H = (H_{\mathbb{Z}}, Q, \mathcal{F})$$

$H_{\mathbb{Z}}$: a locally constant sheaf of free abelian group of rk2

$$( S = \cup S_\lambda \quad H_{\mathbb{Z}}|S_\lambda \cong \mathbb{Z}_{S_\lambda}^{\oplus 2} )$$

$Q$: a non-singular skew symm. bilinear form $H_{\mathbb{Z}} \times H_{\mathbb{Z}} \to \mathbb{Z}_S$

$\mathcal{F} \subsetneq H_{\mathbb{Z}} \otimes_{\mathbb{Z}_S} \mathcal{O}_S (=: \mathcal{H})$ subbundle of rank 1

( $\mathcal{F}$ and $\mathcal{H}/\mathcal{F}$ are locally free $\mathcal{O}_S$-modules )

such that $(H_{\mathbb{Z},s}, Q_s, \mathcal{F}(s))$ is a pol HS for any $s \in S$

$H_{\mathbb{Z},s}$ : stalk at $s$ $\quad Q_s$: stalk at $s$

$\mathcal{F}(s) = \mathcal{F}_s \otimes_{\mathcal{O}_{S,s}} \mathbb{C}(s)$ : fiber of the line bundle $\mathcal{F}$

Rem. The Griffiths transversality condition: $\nabla \mathcal{F}^p \subseteq \mathcal{F}^{p-1} \otimes \Omega^1_S$
is automatically satisfied in case wt $= 1$.
( $\nabla : \mathcal{H} \to \mathcal{H} \otimes \Omega^1_S$ integrable connection assoc. to $H_{\mathbb{Z}}$ )

"A smooth elliptic fibration defines a pol VHS wt 1 rk 2":

$X \xrightarrow{\pi} S$ smooth elliptic fib

$$H_{\mathbb{Z}} = R^1\pi_* \mathbb{Z}_X$$

$$Q : R^1\pi_*\mathbb{Z}_X \times R^1\pi_*\mathbb{Z}_X \xrightarrow{\cup} R^2\pi_*\mathbb{Z}_X \xrightarrow[\sim]{\text{trace map}} \mathbb{Z}_S$$

$$\mathcal{F} = \pi_*\Omega^1_{X/S} \underset{\Uparrow}{\subseteq} \mathcal{H} (= H_{\mathbb{Z}} \otimes \mathcal{O}_S)$$

$$(R^1\pi_*\mathbb{Z}_X) \otimes_{\mathbb{Z}_S} \mathcal{O}_S \cong R^1\pi_*(\pi^{-1}\mathcal{O}_S)$$

$$0 \to \pi^{-1}\mathcal{O}_S \to \mathcal{O}_X \xrightarrow{d_{X/S}} \Omega^1_{X/S} \to 0 \quad \text{(rel. Poincaré exact seq.)}$$

$\Rightarrow (H_{\mathbb{Z}}, Q, \mathcal{F})$ pol VHS

"A pol VHS wt 1 rk 2 defines a smooth <u>basic</u> elliptic fib":

$$\mathcal{H}/_{H_{\mathbb{Z}} + \mathcal{F}} \longleftrightarrow \text{geometric object}$$

Set $\mathcal{L}_H := \mathcal{H}/\mathcal{F}$. $\quad \begin{matrix} \mathbb{V} = \mathbb{V}(H) \\ \downarrow \\ S \end{matrix}$ geometric line bundle corresp. to $\mathcal{L}_H$
(sheaf of sections $\cong \mathcal{L}_H$)

$$H_{\mathbb{Z}} \hookrightarrow \mathcal{L}_H$$

$$\left( \begin{matrix} H_{\mathbb{Z},s} \hookrightarrow \mathcal{L}_H \otimes \mathbb{C}(s) \\ \forall s \end{matrix} \right) \Rightarrow {}^{\exists}\, \mathbb{L} \subseteq \mathbb{V}, \quad \mathbb{L} \to S,\ \mathbb{V} \to S$$

local constant system.

$\mathbb{B} = \mathbb{B}(H) := \mathbb{V}/\mathbb{L} \xrightarrow{p} S$ smooth basic elliptic fib

$\left( \begin{array}{l} \bullet \ 0\text{-section of } \mathbb{V} \to S \Rightarrow 0\text{-section of } \mathbb{B} \to S \\ \bullet \ R^1 p_* \mathbb{Z}_{\mathbb{B}} \cong H \text{ as pol VHS.} \end{array} \right.$

[Geometric description of $\mathbb{B} \to S$ (due to Kodaira)]

$\tilde{S} \xrightarrow{\tau} S$ universal covering map.

$\pi_1 := \pi_1(S, s)$ ($s \in S$ a reference point) $\quad \pi_1 \curvearrowright \tilde{S}$ (left action) $\quad S \simeq \pi_1 \backslash \tilde{S}$.

$H_{\mathbb{Z}}$ is determined by a monodromy representation

$$\pi_1 = \pi_1(S, s) \xrightarrow{\rho} \mathrm{Aut}(H_{\mathbb{Z},s}, Q_s) \cong SL(2, \mathbb{Z})$$

consider $\mathbb{Z}^2$ as a right $\pi_1$-module by

$$(m, n)^{\gamma} = (m, n)\rho(\gamma)$$

where $\rho(\gamma) = \begin{pmatrix} a_\gamma & b_\gamma \\ c_\gamma, & d_\gamma \end{pmatrix} \in SL(2, \mathbb{Z})$

For a suitable choice of basis $(e_0, e_1)$ of $\mathbb{Z}^2$,

we have a holomorphic function

$$\omega : \tilde{S} \longrightarrow \mathbb{H} := \{ \operatorname{Im} \zeta > 0 \} \subseteq \mathbb{C}$$

such that

$\left( \begin{array}{l} \star \ \tau^* \mathcal{F} = \mathcal{O}_{\tilde{S}} \cdot (\omega e_1 + e_0) \text{ as a subsheaf of } \tau^* \mathcal{H} = \tau^* H_{\mathbb{Z}} \otimes \mathcal{O}_{\tilde{S}} \cong \mathcal{O}_{\tilde{S}}^{\oplus 2} \\ \star\star \ \omega(\gamma z) = \dfrac{a_\gamma \omega(z) + b_\gamma}{c_\gamma \omega(z) + d_\gamma} \text{ for any } \gamma \in \pi_1. \end{array} \right.$

Rem: (pol VHS wt 1 rk 2) $\longleftrightarrow$ $\begin{pmatrix} \rho: \pi_1 \to SL(2,\mathbb{Z}) \\ \omega: \tilde{S} \to \mathbb{H} \end{pmatrix}$ satisfying **)

## Action of $\pi_1 \ltimes \mathbb{Z}^2$ on $\tilde{S} \times \mathbb{C}$

$\pi_1 \ltimes \mathbb{Z}^2 \ni (\gamma, (m,n))$ $\quad (\gamma_1, (m_1\, n_1)) \cdot (\gamma_2, (m_2\, n_2))$

$$= (\gamma_1 \cdot \gamma_2\,,\ (m_1\, n_1)\rho(\gamma_2) + (m_2\, n_2))$$

$$\begin{array}{ccc} \tilde{S} \times \mathbb{C} & \xrightarrow{\Phi(\gamma,(m,n))} & \tilde{S} \times \mathbb{C} \\ \Psi & & \Psi \\ (z, \zeta) & \longmapsto & \left(\gamma z,\ \dfrac{\zeta + m\,\omega(z) + n}{c_\gamma \omega(z) + d_\gamma}\right) \end{array}$$

This action is properly discontinuous and fixed point free.

$$\begin{array}{ccc} \pi_1 \ltimes \mathbb{Z}^2 \backslash \tilde{S} \times \mathbb{C} & \cong & \mathbb{B}(H) \\ \uparrow & & \uparrow \\ \pi_1 \ltimes 0 \backslash \tilde{S} \times \mathbb{C} & \cong & \mathbb{V}(H) \end{array}$$

Rem A smooth basic elliptic fibration $X \to S$ is isomorphic to $\mathbb{B}(H) \to S$ for the pol VHS $H = (R^1\pi_* \mathbb{Z}_X, \ldots)$. (cf. Weierstrass model)

Rem $\mathbb{B}(H) \to S$ has an $S$-group structure

$\mu: \mathbb{B}(H) \times_S \mathbb{B}(H) \longrightarrow \mathbb{B}(H)$ $\quad (b_1, b_2) \mapsto b_1 + b_2$ fiberwise

$\sigma : S \to \mathbb{B}(H)$ a section of $p : \mathbb{B}(H) \to S$

$\Rightarrow \exists$ translation automorphism

$$\begin{tikzcd} \mathbb{B}(H) \arrow[rr, "\sim"] \arrow[dr] & & \mathbb{B}(H) \arrow[dl] \\ & S & \end{tikzcd} \quad tr(\sigma)$$

$$\left( \mathbb{B}(H) \cong S \times_S \mathbb{B}(H) \xrightarrow{\sigma \times_S id} \mathbb{B}(H) \times_S \mathbb{B}(H) \xrightarrow{\mu} \mathbb{B}(H) \right)$$

If $\varphi : \mathbb{B}(H) \xrightarrow{\sim} \mathbb{B}(H)$ automorphism satisfies

$$\begin{tikzcd} \mathbb{B}(H) \arrow[rr, "\sim"] \arrow[dr, "p"'] & & \mathbb{B}(H) \arrow[dl, "p"] \\ & S & \end{tikzcd}$$

$$\varphi^* = id$$

$$R^1 p_* \mathbb{Z}_{\mathbb{B}(H)} \xrightarrow{\sim} R^1 p_* \mathbb{Z}_{\mathbb{B}(H)},$$

then $\varphi = tr(\sigma)$ for the section $\sigma = \varphi_0$ (0-section).

## §1.3 Smooth elliptic fibrations

We fix a pol VHS $H$ on $S$
(wt 1, rk 2)

A marked smooth elliptic fibration : $(X \xrightarrow{f} S,\ \phi : H(f) \xrightarrow{\sim} H)$

$$H(f) = (R^1 f_* \mathbb{Z}_X, \dots f_*\Omega^1_{X/S}) \underset{\cong}{\xrightarrow{\phi}} H$$

pol VHS — isomorphism as VHS — $\phi$ "marking"

$(X \xrightarrow{f} S, \phi) \underset{isom/S}{\cong} (X' \xrightarrow{f'} S, \phi')$

$\overset{def}{\Longleftrightarrow}$ $\exists \varphi$:

$$\begin{tikzcd} X \arrow[rr, "\cong", "\exists\varphi"'] \arrow[dr, "f"'] & \arrow[d, phantom, "\circlearrowleft"] & X' \arrow[dl, "f'"] \\ & S & \end{tikzcd}$$

s.t.

$$\begin{tikzcd} R^1 f_* \mathbb{Z}_X \arrow[dr] & & R^1 f'_* \mathbb{Z}_{X'} \arrow[ll, "\varphi^*"'] \arrow[dl] \\ & H & \end{tikzcd}$$

Commutative diagram

$$\left[ \begin{tikzcd} H(f) \arrow[dr, "\phi"'] & & H(f') \arrow[ll, "\sim"', "\varphi^*"] \arrow[dl, "\phi'"] \\ & H & \end{tikzcd} \right.$$

Def $\mathcal{G}_H := \mathrm{Coker}\,(H_{\mathbb{Z}} \to \mathcal{L}_H = \mathcal{H}/\mathcal{F}_1)$

$(0 \to H_{\mathbb{Z}} \to \mathcal{L}_H \to \mathcal{G}_H \to 0$ exact$)$

Rem $\mathcal{G}_H \cong$ the sheaf of sections of $\mathbb{B}(H) \xrightarrow{p} S$.

<u>Proposition</u>

$\{$marked smooth elliptic fib$\}/\sim_{\text{isom}/S} \underset{1-1}{\Longleftrightarrow} H^1(S, \mathcal{G}_H)$.

$(\to)$: $(X \xrightarrow{f} S, \phi)$ a marked smooth elliptic fib.

$\Rightarrow$ $0 \to \mathbb{Z}_X \to \mathcal{O}_X \to \mathcal{O}_X^\times \to 1$ exponential exact seq.

$u \mapsto \exp(2\pi\sqrt{-1}\cdot u)$

$\Downarrow$

$$\begin{array}{ccccccccc} 0 \to & R^1f_*\mathbb{Z}_X & \to & R^1f_*\mathcal{O}_X & \to & R^1f_*\mathcal{O}_X^\times & \to & R^2f_*\mathbb{Z}_X & \to 0 = R^2f_*\mathcal{O}_X \\ & \wr\| & & \wr\| & & & & \wr\| \text{ trace map} & \\ & H_{\mathbb{Z}} & \to & \mathcal{L}_H & & & & \mathbb{Z}_S & \end{array}$$

$\Longrightarrow$ $0 \to \mathcal{G}_H \to R^1f_*\mathcal{O}_X^\times \to \mathbb{Z}_S \to 0$

extension class = image of $\underset{\substack{\cap \\ H^0(S, \mathbb{Z}_S)}}{1} \longmapsto \underset{\substack{\cap \\ H^1(S, \mathcal{G}_H)}}{\eta} = \eta(X/S, \phi)$.

$H^0(S, \mathbb{Z}_S) \longrightarrow H^1(S, \mathcal{G}_H)$

Another description of $(\to)$:

$X \xrightarrow{f} S$ "smooth" $\Longrightarrow$ $\exists$ open covering $\{S_\lambda\}$ of $S$

$\exists$ $S_\lambda \xrightarrow{\sigma_\lambda} X|_{S_\lambda} = f^{-1}(S_\lambda)$

section of $X|_{S_\lambda} \xrightarrow{f} S_\lambda$.

$$\rightsquigarrow \begin{tikzcd} X|_{S_\lambda} \arrow[rr, "\varphi_\lambda", "\sim"', "\text{isom}"'] \arrow[dr, "f"'] & \circlearrowleft & B|_{S_\lambda} \arrow[dl, "p"] \\ & S_\lambda & \end{tikzcd}$$

By taking a suitable automorphism $\begin{tikzcd} B|_{S_\lambda} \arrow[rr, "\sim"] \arrow[dr] & & B|_{S_\lambda} \arrow[dl] \\ & S_\lambda & \end{tikzcd}$ (preserving the 0-section)

we may assume :

$$R^1 f_* \mathbb{Z}_X|_{S_\lambda} \xleftarrow{\varphi_\lambda^*} R^1 p_* \mathbb{Z}_B|_{S_\lambda} \cong H \quad \text{coincides with } \Phi^{-1}|_{S_\lambda}.$$

~~is isom~~

$$\Rightarrow \varphi_\lambda \circ \varphi_\mu^{-1} : B|_{S_\lambda \cap S_\mu} \longrightarrow B|_{S_\lambda \cap S_\mu}$$

$$\| \qquad\qquad \sigma_{\lambda,\mu} : S_\lambda \cap S_\mu \longrightarrow B|_{S_\lambda \cap S_\mu} \quad \text{section of } p.$$

$$tr(\sigma_{\lambda,\mu})$$

Here $\sigma_{\lambda,\mu} + \sigma_{\mu,\nu} + \sigma_{\nu,\mu} = 0$ on $S_\lambda \cap S_\mu \cap S_\nu$

$\Rightarrow \{\sigma_{\lambda,\mu}\} \in Z^1(\{S_\lambda\}, \mathcal{O}_H)$ Čech cocycle

$$\rightsquigarrow \eta \in H^1(S, \mathcal{O}_H)$$

$(\Leftarrow)$ conversely for $\eta = \{\sigma_{\lambda,\mu}\}$, we can glue

$\{B|_{S_\lambda}\}$ by $tr(\sigma_{\lambda\mu})$, and get $X = \bigcup (B|_{S_\lambda}) \xrightarrow{f} S$ $\square$

Rem

$X \xrightarrow{f} S$ is a "torsor" of $B \to S$ :

$B \overset{\exists}{\curvearrowright} X$ $\qquad B \times_S X \longrightarrow X \times_S X \Rightarrow$ isom$/S$.

left action over $S$ $\qquad (b, x) \longmapsto b+x, x$

$\{\text{torsor}\} \Leftrightarrow H^1(S, (\text{sheaf of sections}))$

Prop. Let $(X \xrightarrow{f} S, \phi)$ be a marked smooth elliptic fibration and set $\eta = \eta(X/S, \phi) \in H^1(S, \mathcal{O}_H)$.

Then the following 3 conditions are equiv:

(i) There is a prime divisor $D \subseteq X$ s.t. $f(D) = S$

(ii) $f$ is a projective morphism, i.e., $\exists$ an $f$-ample invertible sheaf

(iii) $\eta \in H^1(S, \mathcal{O}_H)_{tor}$ (torsion element).

# §2 Weierstrass models

## §2.1 Definition

$S$ : a complex manifold. $\mathcal{L} \in \mathrm{Pic}(S)$. $a \in H^0(S, \mathcal{L}^{-4})$
$b \in H^0(S, \mathcal{L}^{-6})$

such that $4a^3 + 27b^2 \neq 0$ as an element of $H^0(S, \mathcal{L}^{-12})$.

$W(\mathcal{L}, a, b)$ : the Weierstrass model is defined to be a divisor of $\mathbb{P} = \mathbb{P}_S(\mathcal{O} \oplus \mathcal{L}^2 \oplus \mathcal{L}^3)$ as follows:

$$\begin{array}{c} \mathbb{P} \\ \downarrow p \\ S \end{array} : \mathbb{P}^2\text{-bundle, where } p_*\mathcal{O}_{\mathbb{P}}(1) \cong \mathcal{O} \oplus \mathcal{L}^2 \oplus \mathcal{L}^3.$$

$X \in H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(1) \otimes p^*\mathcal{L}^{-2})$, $Y \in \mathcal{O}_{\mathbb{P}}(1) \otimes p^*\mathcal{L}^{-3}$, $Z \in \mathcal{O}_{\mathbb{P}}(1)$

corresp to: $\mathcal{L}^2 \hookrightarrow \underbrace{\mathcal{O} \oplus \mathcal{L}^2 \oplus \mathcal{L}^3}_{= p_*\mathcal{O}_{\mathbb{P}}(1)}$, $\mathcal{L}^3 \hookrightarrow \mathcal{O} \oplus \mathcal{L}^2 \oplus \mathcal{L}^3$, $\mathcal{O} \hookrightarrow \mathcal{O} \oplus \mathcal{L}^2 \oplus \mathcal{L}^3$

(canonical injection)

$W(\mathcal{L}, a, b)$ = divisor of zeros of
$$Y^2Z - (X^3 + aXZ^2 + bZ^3) \in H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(3) \otimes p^*\mathcal{L}^{-6})$$

<u>Properties</u> (1) $p : W = W(\mathcal{L}, a, b) \to S$ is proper and flat.

Here, · $\forall$ fiber is an irreducible plane cubic curve.

· $s \notin \{4a^3 + 27b^2 = 0\} \iff p^{-1}(s)$ is non-singular.

(2) $W$ is normal and Gorenstein,

$$\omega_W \cong p^*(\omega_S \otimes \mathcal{L}^{-1}).$$

(3) the 3rd projection $\mathcal{O} \oplus \mathcal{L}^2 \oplus \mathcal{L}^3 \twoheadrightarrow \mathcal{L}^3$ defines a section

$$\begin{tikzcd} \Sigma = \Sigma(\mathcal{L}, a, b) \arrow[r, phantom, "\subseteq"] \arrow[dr, "\sim"'] & \mathbb{P}(\mathcal{O} \oplus \mathcal{L}^2 \oplus \mathcal{L}^3) \arrow[d, "p"] \\ & S \end{tikzcd} \quad \Bigg| \quad \Sigma = (X = Z = 0).$$

Then, $\cdot$ $\Sigma \subseteq W$,

$\cdot$ $\Sigma$ is a Cartier divisor of $W$

$\cdot$ $p : W \to S$ is smooth around $\Sigma$.

(4) $R^1 p_* \mathcal{O}_W \cong \mathcal{L}$.

<u>Def</u>. $(\mathcal{L}, a, b)$ is called minimal if there exist no non-zero ~~effective~~ effective divisor $\Delta$ such that

$$\underbrace{\mathrm{div}(a)}_{\text{(divisor of zeros of } a)} \geq 4\Delta \quad \text{and} \quad \underline{\mathrm{div}(b)} \geq 6\Delta$$

If $(\mathcal{L}, a, b)$ is not minimal, then $\exists!$ maximal non-zero effective divisor $\Delta$ such that $\mathrm{div}(a) \geq 4\Delta$, $\mathrm{div}(b) \geq 6\Delta$,

~~and $(\mathcal{L}(\Delta))$~~

Thus. $a = \delta^4 a_0$, $b = \delta^6 b_0$ for some $\delta_{\neq 0} \in H^0(S, \mathcal{O}(\Delta))$

and $a_0 \in H^0(\mathcal{L}(\Delta)^{-4})$, $b_0 \in H^0(\mathcal{L}(\Delta)^{-6})$, where $\mathcal{L}(\Delta) = \mathcal{L} \otimes \mathcal{O}_S(\Delta)$.

Here $(\mathcal{L}(\Delta), a_0, b_0)$ is minimal.

There is a bimeromorphic map $W(\mathcal{L}, a, b) \dashrightarrow W(\mathcal{L}(\Delta), a_0, b_0)$ over $S$

as follows:

Now: $Y^2 Z = X^3 + a_0 \delta^4 X Z^2 + b_0 \delta^6 Z^3$

$$\Updownarrow$$

$$(\delta^{-3} Y)^2 \cdot Z = (\delta^{-2} X)^3 + a_0 (\delta^{-2} X) Z^2 + b_0 Z^3$$

$(X : Y : Z) \longmapsto (\delta^{-2} X : \delta^{-3} Y : Z)$ induces

a bimeromorphic maps $\mathbb{P}(\mathcal{O} \oplus \mathcal{L}(\Delta)^2 \oplus \mathcal{L}(\Delta)^3) \dashrightarrow \mathbb{P}(\mathcal{O} \oplus \mathcal{L}^2 \oplus \mathcal{L}^3)$

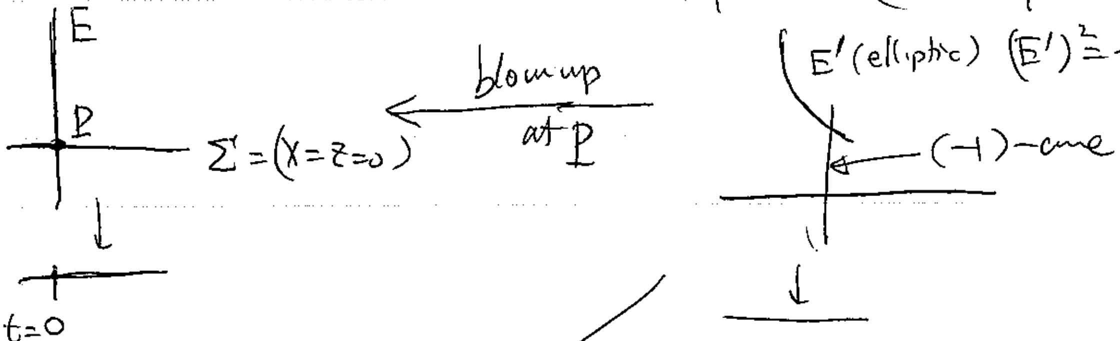
and

$$\begin{array}{ccc} \mathbb{P}(\mathcal{O} \oplus \mathcal{L}(\Delta)^2 \oplus \mathcal{L}(\Delta)^3) & \dashrightarrow & \mathbb{P}(\mathcal{O} \oplus \mathcal{L}^2 \oplus \mathcal{L}^3) \\ \cup & & \cup \\ W(\mathcal{L}(\Delta), a_0, b_0) & \dashrightarrow & W(\mathcal{L}, a, b). \end{array}$$

Example. $E$: elliptic curve : $Y^2 Z = X^3 + \alpha X Z^2 + \beta Z^3$

$\alpha, \beta \in \mathbb{C}$ $(4\alpha^3 + 27\beta^2 \neq 0)$.

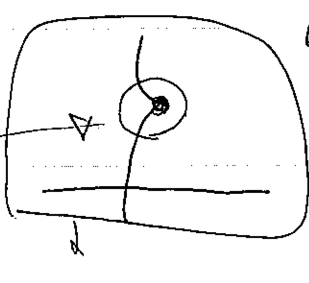
$$\begin{array}{c} E \times \mathbb{C} \\ \downarrow \\ \mathbb{C} \end{array}$$




contraction of $E'$

$E \times \mathbb{C}$

$= W(\mathcal{O}, \alpha, \beta)$



Simple elliptic sing.

$= W(\mathcal{O}, \alpha t^4, \beta t^6)$

$Y^2 Z = X^3 + \alpha t^4 X Z^2 + \beta t^6 Z^3$

## §2.2. Theorems.

Thm 1 (Morphism to Weierstrass model)

Let $\pi : X \longrightarrow S$ be an elliptic fibration with a section $\sigma : S \to X$.

Assume that $X$ and $S$ are non-singular.

Then $\exists$ a Weierstrass model $W = W(\mathcal{L}, a, b)$ over $S$

$$\exists\ \mu : X \longrightarrow W \quad \text{bimeromorphic morphism}$$

$$\begin{tikzcd} X \arrow[rr, "\mu"] \arrow[dr, "\pi"'] & & W \arrow[dl, "p"] \\ & S & \end{tikzcd}$$

Such that

- $\sigma(S) = \mu^* \Sigma(\mathcal{L}, a, b)$ (as a divisor)
- If $\pi$ is smooth over an open subset $U \subseteq S$, then $\{4a^3 + 27b^2 = 0\} \cap U = \phi$.

Rem. Assume: $\pi$ is smooth in Th 1. (i.e., smooth basic elliptic fib).

Then $X \xrightarrow[\cong]{\mu} W(\mathcal{L}, a, b)$ isom.

Here $\mathcal{L} \cong \mathcal{L}_H$

$a = \text{const.}\ G_4(w(z))$

$b = \text{const.}\ G_6(w(z))$

$$12^3 \frac{4a^3}{4a^3 + 27b^2} = j(w(z)).$$

In particular

$$(\text{smooth basic ell fib}) \longleftrightarrow \begin{pmatrix} \text{pol VHS} \\ \text{wt 1 rk 2} \end{pmatrix} \Longleftrightarrow \left( \underset{4a^3+27b^2 \text{ nowhere vanish.}}{(\mathcal{L}, a, b)} \right)$$

Cor of Thm 1. Let $p: W = W(\mathcal{L}, a, b) \to S$ be a Weierstrass model over $S$ (manifold).

Assume that $(4a^3 + 27b^2 = 0)$ is a normal crossing divisor.

Then

$(\mathcal{L}, a, b)$ is minimal if and only if $W$ has only rational (Gorenstein) singularities.

(rat. Gor. sing = canonical singularity of index 1)

In this case,

$$\mathcal{L} \cong \mathcal{L}_{H/S} \ (:= \mathrm{Gr}^0_{\mathcal{F}}({}^{\ell}\mathcal{H}_S))$$

lower canonical extension of $\mathcal{H} = H_{\mathbb{Z}} \otimes \mathcal{O}_{S \setminus D}$

$D = (4a^3 + 27b^2 = 0)$.

"canonical extension in the sense of Deligne"

(cf. Deligne: Springer Lecture Notes in Math 163 (1970)
Schmid: Variation of Hodge str. Invent. Math 22 (1973)
Kollár: Higher direct images -- II, Ann. Math. 124 (1986))

${}^{\ell}\mathcal{H}_S$ is a locally free $\mathcal{O}_S$ module of rk 2 $\subseteq j_*\mathcal{H}$

s.t. ${}^{\ell}\mathcal{H}_S|_{S \setminus D} \cong \mathcal{H}$ $\qquad j: S \setminus D \hookrightarrow S$ open immersion

$+ \ldots$

$\mathcal{F}^1({}^{\ell}\mathcal{H}_S) := j_*\mathcal{F} \cap {}^{\ell}\mathcal{H}_S \subseteq j_*\mathcal{H}$

also a subbundle of ${}^{\ell}\mathcal{H}_S$ (a special case of Schmid's nilpotent orbit theorem)

Thm 2 (Extension)

$S$ manifold $\supseteq U$ Zariski open dense subset.

$W_U = W(\mathcal{L}_U, a_U, b_U) \longrightarrow U$ smooth Weierstrass model ($4a_U^3 + 27b_U^2$ is nowhere vanishing)

$\Rightarrow \exists (\mathcal{L}, a, b)$ minimal s.t.

$$a|_U = \varepsilon^4 a_U \text{ and } b|_U = \varepsilon^6 b_U$$

for a nowhere vanishing section $\varepsilon \in H^0(U, \mathcal{L}_U \otimes (\mathcal{L}|_U)^{-1})$

Moreover $(\mathcal{L}, a, b)$ above is "unique":

If $(\mathcal{L}', a', b')$ satisfies the above condition, then

$a = e^4 a'$ and $b = e^6 b'$ for some nowhere vanishing section $e \in H^0(S, \mathcal{L}' \otimes \mathcal{L}^{-1})$.

In particular, $W_U \to U$ extends uniquely to a <u>minimal</u> Weierstrass model over $S$

(Rem: If $S \setminus U$ is normal crossing, then $\mathcal{L} \cong Gr^0_{\mathcal{F}}(\mathcal{H}_S)$ $(=: \mathcal{L}_{H/S})$.)

<u>Cor.</u> Let $H$ be a pol VHS wt 1, rk 2 defined on Zariski open dense subset $U \subseteq S$.

Then $\exists$ $p: B \longrightarrow S$ elliptic fibration with a section s.t. $p^{-1}(U) \to U$ smooth, $H(p|_{p^{-1}U}) \cong H$ as pol VHS.

Here $p: B \to S$ is unique up to bimeromorphic equiv. rel over $S$.

i.e., if $p': B' \to S$ satisfies the same condition, then

there exists a meromorphic map
$$\begin{tikzcd} B \arrow[rr, dashed, "\varphi"] \arrow[dr, "p"'] & & B' \arrow[dl, "p'"] \\ & S & \end{tikzcd}$$

such that $\varphi|_{p^{-1}U} : p^{-1}U \to p'^{-1}U$ is an isomorphism $\square$

$$\left(\begin{array}{l}\text{pol VHS} \\ \text{wt 1 rk 2 on } U\end{array}\right) \longleftrightarrow \left(\begin{array}{l}(\mathcal{L}, a, b) \text{ minimal on } S \\ \text{s.t. } (4a^3+27b^2=0) \cap U = \phi\end{array}\right).$$

Cor. Let $S$ be a complex manifold, $D \subseteq S$ a nonsingular divisor.

Let $H$ be a pol VHS wt 1 rk 2 on $U := S \setminus D$.

Then $\exists ! \; B \xrightarrow{p} S$ elliptic fibration ~~with~~ such that
(unique up to isom/$S$)

1) $B$ is nonsingular
2) $K_B \sim p^*(K_S - \mathcal{L}_{H/S})$ $\left( \omega_B \cong p^*(\omega_S \otimes \mathcal{L}_{H/S}^{-1}) \right)$
3) $p$ admits a section
4) $p: B \to S$ is flat.

$\because$) $W = W(\mathcal{L}_{H/S}, a, b) \to S$ Weierstrass model

Sing $W$ : locally (Rational Double Pts) $\times \mathbb{C}^{d-1}$
$d = \dim S$.

$\therefore$ We have a "minimal resolution" of $W$ $(B \to W)$

by the minimal resolution of the rational double pts.

Then $B \to S$ satisfies all the conditions 1) – 4).

--- □

Set $B^\# = \{ b \in B \mid p : B \to S$ is smooth at $b \}$.

Then $B^\# \to S$ has an $S$-group structure.

extending $B/_{S \setminus D} \to S \setminus D$

$B^\#/S$ is called the analytic Néron model.

$B^\# \curvearrowright B$
acts

<u>Def</u> $S$ manifold, $D$ ~~non-singular~~ normal crossing divisor,

$H$ polVHS wt 1 rk 2 on $S \setminus D$.

<u>A marked elliptic fibration</u>: $(X \xrightarrow{f} S, \phi)$ w.r.t. $(S, D, H)$

$f : X \to S$ elliptic fibration smooth over $S \setminus D$

$\phi : H(f) = (R^1 f_* \mathbb{Z}_X /_{S \setminus D}, \ldots) \xrightarrow{\sim} H$ isom as polVHS.

<u>Def</u> $\mathcal{E}_0(S, D, H) = \{$ A marked elliptic fibration $(X \xrightarrow{f} S, \phi) \mid$

$f$ admits meromorphic sections locally on $S \} / \sim_{\text{bim}/S}$.

$(S = \bigcup S_\lambda$ $\quad S_\lambda \dashrightarrow f^{-1}(S_\lambda)$ meromorphic section$)$

- a meromorphic section $S \dashrightarrow X$ corresp. to a prime divisor $D \subseteq X$ such that

$$\begin{tikzcd} D \arrow[r, phantom, "\subseteq"] \arrow[dr] & X \arrow[d] \\ & S \end{tikzcd}$$

bimeromorphic morphism.

- $\sim_{\mathrm{bim}/S}$ $\quad (X \xrightarrow{f} S, \phi) \sim_{\mathrm{bim}/S} (X' \xrightarrow{f'} S, \phi')$

$\Longleftrightarrow$ $\exists$

$$\begin{tikzcd} X \arrow[rr, dashed, "\varphi"] \arrow[dr, "f"'] & & X' \arrow[dl, "f'"] \\ & S & \end{tikzcd} \quad \text{s.t} \quad \begin{tikzcd} H(f) \arrow[dr, "\phi"'] & & H(f') \arrow[ll, "\varphi^*"', "\cong"] \arrow[dl, "\phi'"] \\ & H & \end{tikzcd}$$

bimeromorphic map

$$\left( \Rightarrow \begin{tikzcd} f^{-1}U \arrow[rr, "\sim"] \arrow[dr] & & f'^{-1}U \arrow[dl] \\ & U & \end{tikzcd} \text{(isom)} \right)$$

Thm. If $D$ is nonsingular, then (cf. Kodaira: $\dim S = 1$ On complex analytic surfaces II Ann. Math. 77 (1963))

$$\mathcal{E}_0(S, D, H) \longleftrightarrow H^1(S, \mathcal{G}_{H/S}),$$

where $\mathcal{G}_{H/S}$ = the sheaf of sections of $B \to S$
$\qquad (=$ sections of $B^\# \to S)$.

Rem. $j : S \setminus D \hookrightarrow S$ open immersion

$$0 \to H \to \mathcal{L}_H \to \mathcal{G}_H \to 0 \quad \text{exact on } S \setminus D$$

$$\Rightarrow \left[ \begin{tikzcd} 0 \arrow[r] & j_*H \arrow[r] & j_*\mathcal{L}_H \arrow[r] & j_*\mathcal{G}_H \arrow[r] & R^1 j_* H \arrow[r] & 0 = R^1 j_* \mathcal{L}_H \\ 0 \arrow[r] & j_*H \arrow[r] \arrow[u, equal] & \mathcal{L}_{H/S} \arrow[r] \arrow[u, phantom, "\cup"] & \mathcal{G}_{H/S} \arrow[r] \arrow[u, phantom, "\cup"] & (R^1 j_* H)_{\mathrm{tor}} \arrow[r] \arrow[u, phantom, "\cup"] & 0 \end{tikzcd} \right.$$

$(X \xrightarrow{f} S, \phi) \in \mathcal{E}_0(S, D, H)$

cf *) Kollár's torsion freeness Ann Math 124

$\Rightarrow$

$$\begin{tikzcd}
0 \arrow[r] & R^1 f_* \mathbb{Z}_X \arrow[r] \arrow[d, "s"] & R^1 f_* \mathcal{O}_X \arrow[r] \arrow[d, equal, "s\ *)"] & R^1 f_* \mathcal{O}_X^\times \arrow[r] & R^2 f_* \mathbb{Z}_X \arrow[r] & R^2 f_* \mathcal{O}_X \arrow[r] \arrow[d, equal, "*)"] & {} \\
& J_* H \arrow[r] & \mathcal{L}_{H/S} & & & 0 &
\end{tikzcd}$$

$$\mathcal{V}_X := \underbrace{\mathcal{H}^0_D}_{\text{local cohomology}}(R^1 f_* \mathcal{O}_X^\times) = \mathrm{Ker}\left(R^1 f_* \mathcal{O}_X^\times \to j_*(R^1 f_* \mathcal{O}_X^\times|_{S \setminus D})\right)$$

$\Rightarrow$ We have

$$0 \to \mathcal{G}_{H/S} \to R^1 f_* \mathcal{O}_X^\times / \mathcal{V}_X \to \mathbb{Z}_S \to 0 \quad \text{exact}$$

$$\Rightarrow \eta(X/S, \phi) \in H^1(S, \mathcal{G}_{H/S}) \quad \text{extension class. } \square$$

~~Gene~~ For $\eta \in H^1(S, \mathcal{G}_{H/S})$

Using the action $\mathcal{B}^\# \curvearrowright B$, we have $B^\eta = \bigcup (B/S_\lambda)$ gluing via $\eta$.

$B^\eta \xrightarrow{f} S$ : nonsingular relatively <u>minimal model</u> over $S$

($K_{B^\eta} \sim f^*$(line bundle))

§4 ..

$S$ : complex manifold  $D$ : normal crossing divisor.

$\mathcal{E}^{proj}(S, D, H) = \{$ marked elliptic fibration $(X \xrightarrow{f} S, \phi)$ s.t.

$f$ is birat. to a projective morphism $\} / \sim_{bir/S}$

Then $\mathcal{E}^{proj}(S, D, H) \longleftrightarrow H^1(\underline{S}, \mathcal{G}_{H/\underline{S}})_{tor}$
(torsion part)

$\underline{S} = (S, D)$ : the $\partial$-space assoc. to $S$ and $D$
with the $\partial$-étale site.

$\mathcal{G}_{H/\underline{S}}$ : the sheaf (in the $\partial$-étale topology) of
$\underbrace{\text{local}}_{\text{meromorphic}}$ sections of $\underset{\substack{\text{bir} \\ W(\mathcal{L}, a, b)}}{B} \longrightarrow S$.

(*) $(U, \Delta) \to (S, D)$ is $\partial$-étale iff $\left( \begin{array}{l} \cdot\ U \xrightarrow{\lambda} S \text{ has only discrete fibers} \\ \cdot\ \lambda^{-1}(D) \subseteq \Delta \\ \cdot\ U \setminus \Delta \to S \setminus D \text{ is étale} \end{array} \right)$

[ "Global str" Publ RIMS. ]

§3. $S = \Delta^d$ d-dim polydisc $(t_1, \dots, t_d)$ $|t_i| < 1$

$D = (t_1 \cdots t_l = 0)$  $1 \le l \le d$.

$H$ pol VHS wt 1 rk 2 on $S \setminus D \cong (\Delta^\times)^\ell \times \Delta^{d-\ell}$

$\pi_1 \cong \mathbb{Z}^\ell$

Type of $H$: $\rho : \pi_1(S \setminus D) \longrightarrow SL(2, \mathbb{Z})$ monodromy rep.

(1) Suppose $\mathrm{Image}(\rho)$ is a finite gp.

$H$ is of type $I_0 \underset{\mathrm{def}}{\Leftrightarrow} \mathrm{Im}(\rho) = \{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\}$.

type $I_0^{(*)} \underset{\mathrm{def}}{\Leftrightarrow} \mathrm{Im}\,\rho \cong \mathbb{Z}/2\mathbb{Z} \Leftrightarrow \#\mathrm{Im}\,\rho = 2$

$II^{(*)} \underset{\mathrm{def}}{\Leftrightarrow} \mathrm{Im}\,\rho \cong \mathbb{Z}/6\mathbb{Z} \Leftrightarrow \#\mathrm{Im}\,\rho = 6$

$III^{(*)} \underset{\mathrm{def}}{\Leftrightarrow} \mathrm{Im}\,\rho \cong \mathbb{Z}/4\mathbb{Z} \Leftrightarrow \#\mathrm{Im}\,\rho = 4$

$IV^{(*)} \underset{\mathrm{def}}{\Leftrightarrow} \mathrm{Im}\,\rho \cong \mathbb{Z}/3\mathbb{Z} \Leftrightarrow \#\mathrm{Im}\,\rho = 3$.

(2) Suppose $\mathrm{Im}(\rho)$ is infinite. Then $\exists P \in SL(2, \mathbb{Z})$

$\exists\, a : \pi_1 \longrightarrow \mathbb{Z}$, $\exists\, c : \pi_1 \longrightarrow \mathbb{Z}/2\mathbb{Z}$ s.t.

$\overset{\#}{\circ}$

$$P^{-1} \rho(\gamma) P = (-1)^{c(\gamma)} \begin{pmatrix} 1 & a(\gamma) \\ 0 & 1 \end{pmatrix}.$$

- $H$ is of Type $I_{(+)}$ (or more precisely $I_a$)

$\underset{\mathrm{def}}{\Leftrightarrow} c = 0$

- $H$ is of Type $I_{(+)}^{(*)} \Leftrightarrow c \neq 0$. In this case

we set $a^* : \pi_1 \to \mathbb{Z}$ by $a^*(\gamma) = (-1)^{c(\gamma)} a(\gamma)$.

Then H is of type $\mathrm{I}_{(+)}^{(*)}(0) \underset{\text{def}}{\Longleftrightarrow} \alpha^* \equiv 0 \mod 2$

$\mathrm{I}_{(+)}^{(*)}(1) \Longleftrightarrow \alpha^* \equiv \mathbb{C} \mod 2$

$\mathrm{I}_{(+)}^{(*)}(2) \Longleftrightarrow \alpha^* \wedge \mathbb{C} \not\equiv 0 \mod 2$.

<u>Theorem</u> $\mathcal{E}^{proj}(S, D, H) \cong H^1(\Sigma, \mathfrak{S}_{H/\Sigma})_{tor}$ is calculated as follows:

Type $\mathrm{I}_0$, $\mathrm{I}_0^{(*)}$ $\mathrm{II}^{(*)}$ $\mathrm{III}^{(*)}$ $\mathrm{IV}^{(*)}$

$(\mathbb{Q}/\mathbb{Z})^{\oplus 2l}$ $(\mathbb{Z}/2\mathbb{Z})^{\oplus 2(l-1)}$ $0$ $(\mathbb{Z}/2\mathbb{Z})^{\oplus(l-1)}$ $(\mathbb{Z}/3\mathbb{Z})^{\oplus(l-1)}$

Type $\mathrm{I}_{(+)}(\mathrm{I}_a)$ $\mathrm{I}_{(+)}^{(*)}(0)$ $\mathrm{I}_{(+)}^{(*)}(1)$ $\mathrm{I}_{(+)}^{(*)}(2)$

$(\mathbb{Q}/\mathbb{Z})^{\oplus(l-k+1)} \oplus \mathbb{Z}/\alpha\mathbb{Z}$ $(\mathbb{Z}/2\mathbb{Z})^{\oplus 2(l-1)}$ $(\mathbb{Z}/4\mathbb{Z})^{\oplus(l-1)}$ $(\mathbb{Z}/2\mathbb{Z})^{\oplus(l-1)}$

$k = \#\{\gamma_i \mid a(\gamma_i) > 0\}$,

$\begin{pmatrix} \gamma_1, \ldots, \gamma_l \in \pi_1(S \setminus D) \\ \text{standard generator} \end{pmatrix}$

$\alpha = \gcd(a(\gamma_1) \ldots a(\gamma_l))$

<u>Cor</u> If H is not of type $\mathrm{I}_0$ nor $\mathrm{I}_{(+)}$, then

$\mathcal{E}^{proj}(S, D, H) \xrightarrow[\text{restriction}]{} \mathcal{E}^{proj}(S \setminus D, \phi, H) \Leftrightarrow H^1(S \setminus D, \mathfrak{S}_H)_{tor}$

is bijective.