On the convergence of Fourier-Jacobi expansion

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Motivation

On the symmetric domain of type IV, Borcherds has constructed automorphic forms by infinite product in his paper *Automorphic forms on $O_{s+2,2}(\mathbb{R})$ and infinite products* in 1995.

In his paper, he set the discrete group $O_{s+2,2}(\mathbb{Z})$. And he gave an open problem: *Extend the methods of this paper to level greater than 1.*

One of the big obstacle on this problem is convergence. To show the convergence of the infinite product, he did very complicated calculation about asymptotic behavior of the Fourier coefficients of modular forms.

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Abstract

The symmetric domain of type IV is a domain defined from an indefinite quadratic space (V, S) of signature $(2, s+2)$.

Now we treat the case that one can separate two hyperbolic plane from V. Namely, we fix a basis of V and denote

$$S := \begin{pmatrix} & & & & 1 \\ & & & 1 & \\ & & -S_0 & & \\ & 1 & & & \\ 1 & & & & \end{pmatrix},$$

where S_0 be an even integral positive definite symmetric matrix with rank s.

The symmetric domain of type IV is a connected component of $\mathcal{H} = \mathrm{P}_{\mathbb{C}} H_S$, where

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The orthogonal group $G := \mathrm{O}(S;\mathbb{R})^+$ acts on $\mathcal{H}$ transitively.

Let Γ be a finite index subgroup of $\mathrm{O}(S;\mathbb{Z}) \cap G$.

A holomorphic function F on H_S is an automorphic form of weight k if F is a homogeneous function of weight k and Γ-invariant.

Namely, for a holomorphic function on $\mathcal{H}$, we can define an automorphic form as a function with corresponded translation formulas. Usually, we set a coordinate of $\mathcal{H}$ by

$$\mathcal{H} \cong \left\{ Z = \begin{pmatrix} \tau \\ z \\ \omega \end{pmatrix} \in \mathbb{C}^{s+2} \;\middle|\; {}^t(\mathrm{Im}\, Z) S_1 (\mathrm{Im}\, Z) > 0,\ \mathrm{Im}\, \tau > 0 \right\},$$

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Each φ_m is a Jacobi form of index m.

$$\mathbb{M}_k \ni F \mapsto \{\varphi_m\} \in \prod_{m=0}^{\infty} \mathbb{J}_{k,m}$$

This is not surjective.
At least, the image has a kind of symmetry.

Determine the image of this map !!

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The simplest case : $s = 1,\ S_0 = (2)$
(Siegel modular forms of degree 2)

$$\Gamma := \mathrm{Sp}(2,\mathbb{Z}) \qquad (\text{ or } \Gamma_0(N)\ )$$

$\mathbb{M}_k$: space of Siegel modular forms of weight k
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Main theorem

Theorem. (The image of Fourier-Jacobi expansion)

These two maps are surjective, hence bijective.

$$\mathrm{FJ} \ : \ \mathbb{M}_k \ni F \mapsto \{\varphi_m\}_{m=0}^{\infty} \in \left(\prod_{m=0}^{\infty} \mathbb{J}_{k,m} \right)^{\mathrm{sym}}$$

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Siegel upper half space of degree 2

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The symplectic group

$$G := \mathrm{Sp}_2(\mathbb{R}) = \left\{ M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{M}_4(\mathbb{R}) \;\middle|\; {}^tMJM = J := \begin{pmatrix} O_2 & -E_2 \\ E_2 & O_2 \end{pmatrix} \right\}$$

acts on $\mathbb{H}_2$ transitively by

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For a holomorphic function $F : \mathbb{H}_2 \to \mathbb{C}$ and $k \in \mathbb{Z}$, define

$$(F|_k M)(Z) := \det(CZ + D)^{-k} F(M\langle Z \rangle).$$

In today's talk, we set $\Gamma := \mathrm{Sp}_2(\mathbb{Z}) := \mathrm{Sp}_2(\mathbb{R}) \cap \mathrm{M}_4(\mathbb{Z})$.

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Definition. (Siegel modular form of degree 2)

We say F is a Siegel modular form of weight k if a holomorphic function F on $\mathbb{H}_2$ satisfies the condition $F = F|_k M$ for any $M \in \Gamma$. We denote the space of all Siegel modular forms of weight k by $\mathbb{M}_k$.

For simplicity, we denote $F(Z) = F(\tau, z, \omega)$.

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Koecher principle

Let $F \in \mathbb{M}_k$. F has a Fourier expansion

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where $q^n := \mathbf{e}(n\tau) := \exp\left(2\pi\sqrt{-1}n\tau\right)$, $\zeta^l := \mathbf{e}(lz)$ and $p^m := \mathbf{e}(m\omega)$.

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Proposition. (Koecher principle)

$a(n,l,m) = 0$ if $4mn - l^2 < 0$ or $m < 0$.

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Proposition. (Koecher principle)

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Definition. (Siegel cusp form)

we say $F \in \mathbb{M}_k$ is a cusp form of weight k if F satisfies the condition $a(n,l,m) = 0$ unless $4mn - l^2 > 0$. We denote the space of all cusp forms of weight k by $\mathbb{M}_k^c$.

Jacobi group

The element $T := \begin{pmatrix} 1&0&0&0\\ 0&1&0&1\\ 0&0&1&0\\ 0&0&0&1 \end{pmatrix}$ induces $F(\tau, z, \omega) \mapsto F(\tau, z, \omega + 1)$.

Define $G^J := \{M \in G \mid M^{-1}TM = T\}$. ($G = \mathrm{Sp}_2(\mathbb{R})$)

If $F : \mathbb{H}_2 \to \mathbb{C}$ has a period 1 with respect to ω, then $F|_k M$ also has a period 1 for any $M \in G^J$.

Let $m \in \mathbb{Z}$ and $\varphi(\tau, z)$ be a holomorphic function on $\mathbb{H} \times \mathbb{C}$. The image of $\varphi(\tau, z)p^m$ by $M \in G^J$ is a product of a holomorphic function on $\mathbb{H} \times \mathbb{C}$ and p^m. ($p^m = \exp\left(2\pi\sqrt{-1}m\omega\right)$)

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Let $m \in \mathbb{Z}$ and $\varphi(\tau, z)$ be a holomorphic function on $\mathbb{H} \times \mathbb{C}$. The image of $\varphi(\tau, z)p^m$ by $M \in G^J$ is a product of a holomorphic function on $\mathbb{H} \times \mathbb{C}$ and p^m. $\quad (\, p^m = \exp\left(2\pi\sqrt{-1}m\omega\right) \,)$

Proposition. (Action of Jacobi group)

For each $m \in \mathbb{Z}$, the group G^J acts on the set of all holomorphic functions on $\mathbb{H} \times \mathbb{C}$.

Jacobi group invariant function

Define $\Gamma^J := G^J \cap \Gamma$. $\quad$ ($\Gamma = \mathrm{Sp}_2(\mathbb{Z})$)

Let $m \in \mathbb{Z}$ and $\varphi(\tau, z)$ be a holomorphic function on $\mathbb{H} \times \mathbb{C}$. We assume that $\varphi(\tau, z)p^m$ is Γ^J-invariant.

Namely, $\varphi(\tau, z)$ satisfies the following two equations:

$$\begin{aligned}\varphi(\tau, z) =& (c\tau + d)^{-k} \mathbf{e}\left(\frac{-mcz^2}{c\tau + d}\right) \varphi\left(\frac{a\tau + b}{c\tau + d}, \frac{z}{c\tau + d}\right) \\ \varphi(\tau, z) =& \mathbf{e}\left(m\left(x^2\tau + 2xz\right)\right) \varphi\left(\tau, z + x\tau + y\right) \\ & \left(\mathbf{e}(x) = \exp\left(2\pi\sqrt{-1}x\right) \right)\end{aligned}$$

for any $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ and for any $x, y \in \mathbb{Z}$.

(c.f. the book by Eichler and Zagier)

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Fourier expansion of Jacobi forms

Above φ has a Fourier expansion

$$\varphi(\tau, z) = \sum_{n,l} c(n,l)\, q^n \zeta^l. \qquad (q^n := \mathbf{e}(n\tau),\ \zeta^l := \mathbf{e}(lz))$$

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If $m > 0$, $c(n,l)$ depends only on $4mn - l^2$ and $l \pmod{2m}$.

Especially, if $m = 1$, $c(n,l)$ depends only on $4mn - l^2$ and sometimes we denote $c(4mn - l^2) = c(n,l)$.

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Jacobi form

Definition. (Jacobi form)

We say above φ is a Jacobi form of weight k and index m if $c(n,l)=0$ except when $n \geq 0$ and $4mn - l^2 \geq 0$.

We denote the space of all Jacobi form of weight k and index m by $\mathbb{J}_{k,m}$.

Jacobi form	$n \geq 0$ and $4mn - l^2 \geq 0$	$\mathbb{J}_{k,m}$
Jacobi cusp form	$n > 0$ and $4mn - l^2 > 0$	$\mathbb{J}^{\mathrm{c}}_{k,m}$
weak Jacobi form	$n \geq 0$	$\mathbb{J}^{\mathrm{w}}_{k,m}$
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Property of Jacobi forms (1)

For $\varphi \in \mathbb{J}^{\mathrm{wh}}_{k,m}$, a positive valued function

$$G_\varphi(\tau, z) := |\varphi(\tau, z)| \exp\left(\frac{-2\pi m(\operatorname{Im} z)^2}{\operatorname{Im} \tau}\right) (\operatorname{Im} \tau)^{\frac{k}{2}}$$

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Proposition. (Upper bound of a Jacobi cusp form)

If $\varphi \in \mathbb{J}^{\mathrm{c}}_{k,m}$, G_φ has a maximum value.

Proposition. (Upper bound of Fourier coefficients)

For $\varphi \in \mathbb{J}^{\mathrm{c}}_{k,m}$, there exists a constant K_φ such that
$|c(n,l)| \leq K_\varphi \left(4mn - l^2\right)^{\frac{k}{2}}$.

$$\left(\quad \varphi(\tau, z) = \sum_{n,l} c(n,l)\, q^n \zeta^l, \qquad (q^n := \mathbf{e}(n\tau),\ \zeta^l := \mathbf{e}(lz)) \quad \right)$$

Property of Jacobi forms (2)

If $m = 0$, φ is a $\mathrm{SL}_2(\mathbb{Z})$-invariant holomorphic function on $\mathbb{H}$. Hence $\mathbb{J}^{\mathrm{w}}_{k,0} = \mathbb{J}_{k,0} = \mathbb{A}_k$ and $\mathbb{J}^{\mathrm{c}}_{k,0} = \{0\}$, where $\mathbb{A}_k$ is a space of all elliptic modular forms of weight k w.r.t. $\mathrm{SL}_2(\mathbb{Z})$.

$\mathbb{J}^{\mathrm{w}}_{*,*} := \bigoplus_{k,m\in\mathbb{Z}} \mathbb{J}^{\mathrm{w}}_{k,m}$ is a graded ring of $\mathbb{A}_* := \bigoplus_{k,m\in\mathbb{Z}} \mathbb{A}_k$.

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Proposition. (Structure theorem of weak Jacobi forms)

$\mathbb{J}^{\mathrm{w}}_{*,*}$ is generated by $\varphi_{0,1}$, $\varphi_{-2,1}$ and $\varphi_{-1,2}$ on $\mathbb{A}_*$.

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Fourier-Jacobi expansion

The Fourier Jacobi expansion is a p-expansion of $F \in \mathbb{M}_k$ or $\mathbb{M}_k^{\mathrm{c}}$:

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Definition. (Fourier-Jacobi expansion)

The Fourier Jacobi expansion is a map from $\mathbb{M}_k$ or $\mathbb{M}_k^{\mathrm{c}}$ to the infinite direct product space of Jacobi forms:

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The symmetry

The element $S := \begin{pmatrix} 0&1&0&0\\ 1&0&0&0\\ 0&0&0&1\\ 0&0&1&0 \end{pmatrix}$ induces $F(\tau, z, \omega) \mapsto (-1)^k F(\omega, z, \tau)$.

$S \in \Gamma$ gives the information about the image of the Fourier-Jacobi expansion. Namely, on the Fourier expansion

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Proposition. (Generators of the symplectic group)

The group $G = \mathrm{Sp}_2(\mathbb{R})$ is generated by G^J and S.
The group $\Gamma = \mathrm{Sp}_2(\mathbb{Z})$ is generated by Γ^J and S.

The image of the Fourier-Jacobi expansion

Let

$$\Big(\prod_{m=0}^{\infty} \mathbb{J}_{k,m}\Big)^{\mathrm{sym}} := \left\{ \{\varphi_m\} \in \prod_{m=0}^{\infty} \mathbb{J}_{k,m} \;\middle|\; c_m(n,l) = (-1)^k c_n(m,l) \right\}$$

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Theorem. (The image of Fourier-Jacobi expansion)

These two maps are surjective, hence bijective.

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Flow of our proof

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- Step 1. Estimation of Jacobi cusp forms on 'rational' points
- Step 2. Convergence at 'rational' points
- Step 3. Complete the proof of FJ^{c}.
- Step 4. Complete the proof of FJ.

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First, we suppose $\{\varphi_m\}_{m=1}^{\infty} \in \left(\prod_{m=1}^{\infty} \mathbb{J}_{k,m}^{\mathrm{c}}\right)^{\mathrm{sym}}$.

Take $x, y \in \mathbb{Q}$ with same denominator D. Then

$$f_m(\tau) := \mathbf{e}(mx^2\tau)\varphi_m(\tau, x\tau + y)$$

is an elliptic cusp form of weight k with respect to the main congruent subgroup of level D^2. The Fourier expansion of f_m is

$$f_m(\tau) = \sum_{\nu>0} a_m(\nu)\, q^{\nu} \qquad \left(a_m(\nu) = \sum_{\substack{n,l\in\mathbb{Z} \\ mx^2+lx+n=\nu}} c_m(n,l) \right).$$

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Because the space of the above elliptic cusp forms is finite dimensional, there exist L and C such that

$$|f_m(\tau)| \leq C \left(\sum_{\nu \leq L} |a_m(\nu)| \right) (\mathrm{Im}\,\tau)^{-\frac{k}{2}}.$$

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Here we denote the Fourier coefficient $c\begin{pmatrix} n & l/2 \\ l/2 & m \end{pmatrix} = c_m(n,l)$.

For any $A \in \mathrm{SL}_2(\mathbb{Z})$, $\begin{pmatrix} {}^tA & O_2 \\ O_2 & A^{-1} \end{pmatrix} \in \Gamma$ induces $c(T) = c({}^tATA)$.

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Proposition. (Structure theorem of weak Jacobi forms)

$\mathbb{J}^{\mathrm{w}}_{*,*}$ is generated by $\varphi_{0,1}$, $\varphi_{-2,1}$ and $\varphi_{-1,2}$ on $\mathbb{A}_*$.

For any $R > 1$ and $(\tau_0, z_0) \in \mathbb{H} \times \mathbb{C}$, there exist its neighbourhood U and $x, y \in \mathbb{Q}$ such that $G_m(\tau, z) \leq R^m C' m^{\frac{k+1}{2}}$ for any $(\tau, z) \in U$. Namely,

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Convergence of Maass lift

Theorem. (Maass Lift)

For any $\varphi \in \mathbb{J}^{c}_{k,1}$, there exists $F \in \mathbb{M}^{c}_{k}$ such that $\mathrm{FJ}^{c}_{1}(F) = \varphi$.

The Hecke operator $T_{-}(m)$ induces a map from $\mathbb{J}^{c}_{k,1}$ to $\mathbb{J}^{c}_{k,m}$.

$$\left(\varphi | T_{-}(m)\right)(\tau, z) := \sum_{ad=m} \sum_{b=0}^{d-1} a^{k} \varphi\left(\frac{a\tau + b}{d}, az\right)$$

The series

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converges by our main theorem.

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Theorem. (Maass Lift)

For any $\varphi \in \mathbb{J}^{\mathrm{c}}_{k,1}$, there exists $F \in \mathbb{M}^{\mathrm{c}}_{k}$ such that $\mathrm{FJ}^{\mathrm{c}}_{1}(F) = \varphi$.

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Lift of a weakly holomorphic Jacobi form

Now let $\varphi(\tau, z) = \sum_{n,l} c(4mn - l^2)\, q^n \zeta^l \in \mathbb{J}^{\mathrm{wh}}_{0,1}$.

Calculate the Maass lift of φ, although it does not converge:

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Hence, **formally**,

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is a Γ^J-invariant function of weight 0. By slight modification, Borcherds constructs a Γ-invariant function of weight $c(0)/2$:

$$p^a\zeta^b q^c \prod_{(m,n,l)>0}\left(1-q^n\zeta^l p^m\right)^{c(4mn-l^2)},$$

where $a=\frac{1}{2}\sum_{l>0} l^2 c(-l^2)$, $b=-\frac{1}{2}\sum_{l>0} lc(-l^2)$, $c=\frac{1}{24}\sum_{l\in\mathbb{Z}} c(-l^2)$ and $(m,n,l)>0$ means $m>0$ or $m=0, n>0$ or $m=n=0, l>0$.

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Convergence

This infinite product (Borcherds product) is a Γ-invariant function and has a symmetry.

However, generally, this infinite product (Borcherds product) does not converge. Borcherds has investigated its analytic continuation. He has determined all zero and poles of this product and shown it to be a meromorphic modular form on $\mathbb{H}_2$.

By our main theorem, if we show each coefficient of the p-expansion of this infinite product, it should be a holomorphic function. Our main theorem holds even when Γ has a level. This gives a partial answer of Borcherds open problem: *Extend the methods of this paper to level greater than* 1.

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Automorphic forms on O(2,s+2)

Generalize our main theorem to automorphic forms on the symmetric domain of type IV:

$$\mathcal{H} := G/K \quad (K = \mathrm{O}(2) \times \mathrm{O}(s+2) : \text{ max. cpt. }) \quad \curvearrowleft \quad \Gamma$$

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Thank you

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