Mordell-Weil Lattices and Elliptic K3 surfaces

at IPMU, 1/6/2010

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Plan.

- MWL (review of basic facts)
- Rational Elliptic Surfaces
- K3 surfaces, with big MW rank
- ell. Integral sections. (no time)
-

Review:

Basic Facts on

Mordell-Weil Lattices

(MWL.)

Key idea:

楕円曲線の有理点のなす群

To view $\boxed{\text{the M-W. group}}$

as a $\boxed{\text{lattice}}$.

$\boxed{1}$ Lattices

- $\boxed{\text{lattice} \overset{\text{def.}}{=} \text{fin. gen. free abelian group} + \text{non-deg. sym. pairing.}}$

$$\begin{cases} L \cong \mathbb{Z}^r & (\text{as abstract groups}) \\ \langle\, , \rangle : L \times L \longrightarrow \mathbb{R}. \end{cases}$$

- pos.-def. lattice $\iff \langle x, x \rangle > 0 \quad \forall x \in L, \neq 0$

L

$\Rightarrow L \subset L \otimes \mathbb{R} = \mathbb{R}^r$

"lattice" in r-dim Euclid. space.

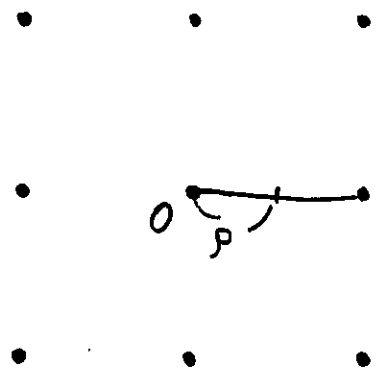
- Such $L \rightsquigarrow$ Sphere packing of $\mathbb{R}^r$

($\rightsquigarrow$ torus $\mathbb{R}^r/L$.

diff. geometry, analysis, ...)

Example. $\overset{r}{\overset{\shortparallel}{n}} = 2$.

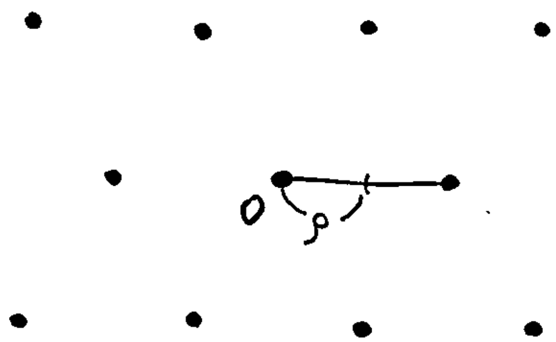
1) $\mathbb{Z}^2 \subset \mathbb{R}^2$ (square lattice)



$\tau = 4$

$\delta = \frac{1}{4}$

2) A_2 ($\sim$ hexagonal lattice



$\tau = 6$

$\delta = \frac{1}{2\sqrt{3}}$.

Basic Invariants of a lattice L.

$r = \operatorname{rank} L$

$\det L$, $\sqrt{\det L} = \operatorname{vol}(\text{fund. dom.})$

$\mu = \underset{x \in L, \neq 0}{\operatorname{Min}} \langle x, x \rangle$, "minimal norm".

min. vector.

$\tau = \#$ min. vectors ("kissing number") density (密度)

$$\Delta = \frac{\mathrm{vol}(1\ \text{sphere})}{\mathrm{vol}(\text{fund. dom. of } L)}$$

center density

$$\delta = \frac{\Delta}{\mathrm{vol}(\text{unit sphere})}$$

Δ, δ: indep. of scaling.

$$= \frac{\rho_L^{\,r}}{\sqrt{\det L}}$$

$$\rho_L = \frac{1}{2}\sqrt{\mu}\ , \ (\text{"packing radius"})$$

<u>Sphere packing problem</u> (lattice packing)

Given $r = \dim$, find L with

Δ (or δ) as large as possible

<u>Kissing number problem</u>

Given r, find L with

τ as large as possible.

Ex. $r=2$. A_2 best for both problems.

Record Lattices, best for S.P.P., K.N.P.

$r = 2$	A_2	$\delta = 1/2\sqrt{3}$	$\tau = 6$
3	A_3	$1/4\sqrt{2}$	12
4	D_4	$1/8$	24
5	D_5	$1/8\sqrt{2}$	40
6	E_6	$1/8\sqrt{3}$	72
7	E_7	$1/8$	126
8	E_8	$1/8$	240

(このページ missing. 1/25/04 追記)

(the above are all root lattices)

⋮

24 Λ_{24} (Leech lattice), $\delta = 1$.

⋮

} ← Some record lattices from $\boxed{\text{MWL}}$ (Elkies & Shioda)

Minkowski ∃ th. (non-constructive)

$\boxed{2}$ Mordell-Weil groups / lattices.

E : elliptic curve over K

$$y^2 = x^3 + Ax + B$$

$A, B \in K$, $\Delta = 4A^3 + 27B^2 \neq 0$.

K-rational points (K-有理点)

$P = (x, y) \in E$ & $x, y \in K$.

$$E(K) = \{ P = \underset{\in E}{(x, y)} \mid x, y \in K \} \cup \{ \underset{\substack{\| \\ \text{pt. at } \infty \\ (0:1:0)}}{O} \}$$

abelian group

<u>M-W. Theorem</u> (1920's, 1950's)

$K = \begin{cases} \mathbb{Q} \text{ or number field} \\ \text{or function field} + \text{mild condition on } E \ (*) \end{cases}$

$\Rightarrow E(K)$ <u>fin. gen.</u>, i.e.

$\simeq \mathbb{Z}^r \oplus$ (finite torsion gr.)

"Mordell-Weil group"

We consider the case:

K = fcn. field of a curve C/k

e.g. $K = k(t)$, $\longleftrightarrow$ $C = \underset{t\text{-line}}{\mathbb{P}^1}$

(k can be $\mathbb{Q}$, $\overline{\mathbb{Q}}$, $\mathbb{C}$; $\mathbb{F}_p$, ...)

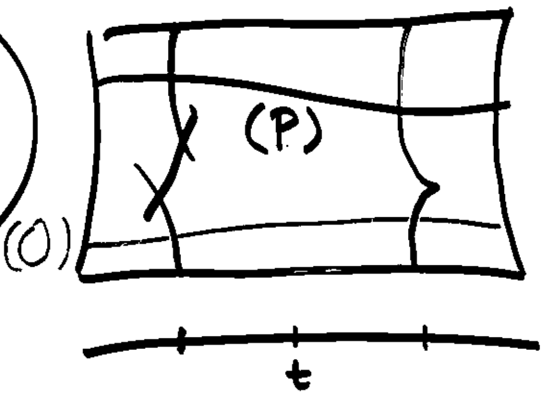
* E/K $\underset{\rightsquigarrow}{\overset{\text{gen. fibre}}{\leftsquigarrow}}$ Elliptic surface

e.g. E: $y^2 = x^3 + t^4 + 1$

P: $\begin{cases} x = -1 \\ y = t^2 \end{cases}$

E | P

Kodaira-Néron model



S smooth projective surface

$\downarrow f$ rel. minimal

C

* K

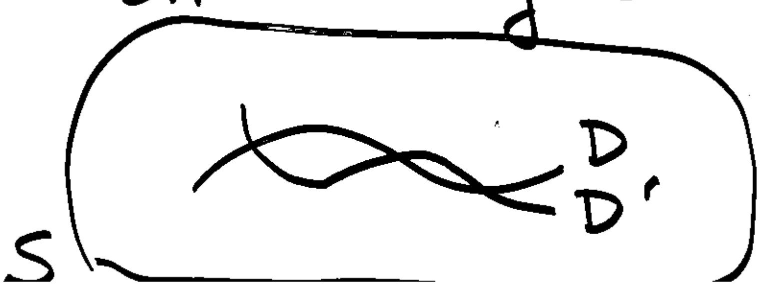
$E(K) \ni P$ $\underset{\text{identify}}{\overset{1:1}{\longleftrightarrow}}$ section of f,
K-rat. pt.

$(P) \subset S$, curve

Basic tool

= Intersection Theory

on the algebraic surface S.



交点数

$(D \cdot D')$ = intersect. number

Assume always that

(*) $\begin{cases} \text{0) } k \text{ : algebraically closed.} \\ \text{i) } f : S \to C \text{ has } 0\text{-section.} \\ \underline{\underline{\text{ii)}}} \quad \text{" has at least one singular fibre.} \end{cases}$

Then

Th. 1) $E(K)$ fin. gen. (M.W. Th.)

2) $NS(S)$ Néron-Severi group of S

$\underset{\text{def.}}{=} \{ \text{divisors } \sum n_i \Gamma_i \} / \text{alg. equiv.}$

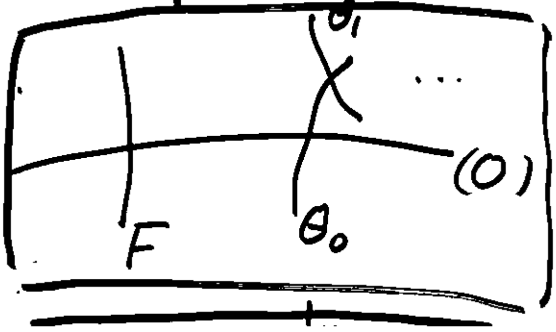
fin. gen. & torsion-free.

3) $$\boxed{\begin{array}{c} E(K) \simeq NS(S)/T \\ P \longleftrightarrow (P) \bmod T \\ \text{natural group isom.} \end{array}} \quad (4)$$

Here "trivial sublattice"

T = generated by (0), F & all irred. components of fibres

a fibre



Cor. $r = \rho - \mathrm{rk}\, T$

$$\boxed{r = \rho - \sum_{v \in P} (m_v - 1) - 2}$$

[So far, no lattice ...

Now the idea of <u>lattice</u> :

- $\{ NS(S)$, intersection pairing $\}$

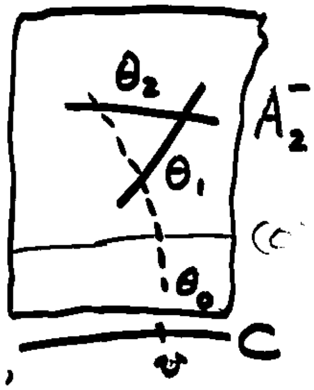
$$(D, D') \longmapsto (D \cdot D') \in \mathbb{Z}$$

is an indefinite lattice

$\uparrow$ signature $(\underset{+}{1}, \underset{-}{\rho - 1})$

[<u>Hodge Index Theorem</u>]

- T is a sublattice



$$= \langle (O), F \rangle \oplus \Big(\bigoplus_{v \in R} \underbrace{T_v}_{\text{neg. def.}} \Big)$$

$R = \{$ reducible sing. fibre $\}$

T_v^- = <u>root lattice of type</u> A, D, E. (← Kodaira, Néron)

Key <u>Lemma 1</u>. $\exists ! \ \varphi : E(K) \longrightarrow NS(S)_{\boxed{\mathbb{Q}}}$

Split (4)

s.t. $\begin{cases} \varphi(P) \equiv (P) \mod T_{\mathbb{Q}} \\ \mathrm{Im}(\varphi) \perp T \end{cases}$

<u>Cor.</u> φ : group homo., $\mathrm{Ker}(\varphi) = E(K)_{tor}$

(Abel's Th. on E/K)

Lemma 2. $L = T^{\perp} \subset NS(S)$

is $\underbrace{\text{neg-def. even}}$ lattice
$\quad$(integral)

now $\varphi : E(K) \longrightarrow L \otimes \mathbb{Q}$.

Theorem-Def.

$P, Q \in E(K)$

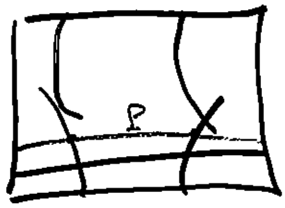
$$\boxed{\langle P, Q \rangle \overset{\text{def.}}{=} -(\varphi(P) \cdot \varphi(Q))} \in \mathbb{Q}$$

$\Rightarrow (E(K)/E(K)_{tor}, \langle\, , \,\rangle)$

is a pos.-def. lattice.

"Mordell-Weil lattice"

(MWL)

Also $\exists\, E(K)^0 \subset E(K)$,  $\circ$

well-defined subgr. fin. index

$(E(K)^0, \langle\, , \,\rangle) \simeq (L^-, -(\, , \,))$

is a pos.-def. even lattice.

"narrow Mordell-Weil lattice"

Explicit formula of the height pairing $\langle \, , \, \rangle$

- P or $Q \in E(K)^0 \Rightarrow$

$$\boxed{\langle P, Q \rangle = \chi + (PO) + (QO) - (PQ)} \in \mathbb{Z}$$

χ: arithmetic genus of S

(PO): int. number of (P) & (O).

esp.

$$\boxed{\langle P, P \rangle = 2\chi + 2(PO)} \geq 2\chi$$

$\geq$ if $P \neq O$.

- $P, Q \in E(K)$

NB. $\langle \, , \, \rangle$ = Néron-Tate height

$$\Rightarrow \boxed{\langle P, Q \rangle = (\text{the above}) - \sum_{v \in R} \mathrm{contr}_v(P, Q)}$$

$\mathrm{contr}_v(P, Q)$: explicitly given rat. numbers ≥ 0.

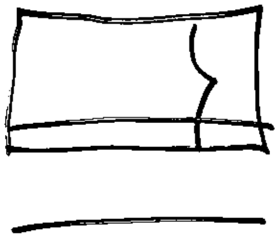
<u>Cor</u>. MWL $\subset$ dual lattice of narrow MWL.

<u>Th</u>. $=$ if $NS(S)$ unimodular.

$\boxed{3}$ First example / case : Rational ell. surface

$\Downarrow$

NS(S) : unimodular

(0)  $S \xrightarrow{f} \mathbb{P}^1$

$\rho(S) = 10$

$\chi = 1$

$K = k(t) \quad \Rightarrow r \leq 8$

Hierarchy E_8 — $E_7 \ldots E_6$

Structure Theorem

(i) $r = 8 \iff$ no reducible fibre

$\Longrightarrow E(K) \simeq E_8$ (root lattice)

(ii) $r = 7 \iff \exists!$ red. fibre I_2 or III

$$\Longrightarrow \begin{array}{ccc} E(K) & \simeq & E_7^* \\ \cup & & \cup \text{ index } 2 \\ E(K)^\circ & \simeq & E_7 \end{array}$$

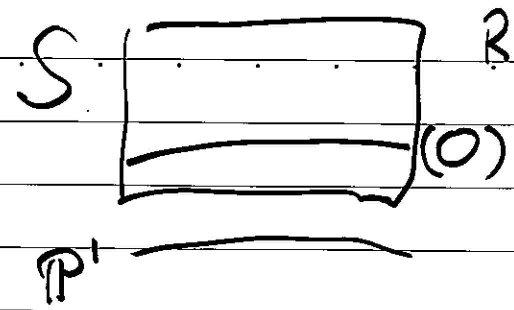
(iii) $r = 6 \iff$ a) $\exists!$ red. fibre I_3 or IV

b) $\exists$ 2 red. fibres I_2 or III

$$\text{a)} \begin{array}{ccc} E(K) & \simeq & E_6^* \\ \cup & & \cup\ 3 \\ E(K)^\circ & \simeq & E_6 \end{array} . \qquad \text{b)} \begin{array}{c} D_6^* \\ \cup\ 4 \\ D_6 \end{array} . \qquad . \ \underline{\text{etc}}$$

+ Generation Theorem

Many Applications — later.



RES. $M = E(K)$: MWL

$L = M^0$: narrow MWL

$T = \bigoplus_{v \in R} T_v^- \hookrightarrow E_8$ root lattice

- all rat. ell. surfaces (with section) are classified into 74 types by $\{T, L, M\}$. (Oguiso–Shioda. 1991)

- $\begin{cases} T: \text{ direct sum of ADE root lattices } \subset E_8 \\ L \cong T^\perp \text{ in } E_8 \\ M \cong \underbrace{L^*}_{\text{dual lattice of } L} \oplus (\text{tor}), \quad (\text{tor}) = T'/T \\ \qquad T \subset T' = \text{primitive closure of } T \text{ in } E_8 \end{cases}$

Rem. the embedding of $T = \bigoplus_v T_v$ (A, D, E) into E_8 is unique except for 5 cases.

Rem. In OS-classification, (coarser than Persson-classif. Miranda) ignored:

$\begin{cases} \cdot \text{ irreducible singular fibres } I_1, \mathrm{II} \longrightarrow T_v = \{0\} \\ \cdot\ I_2 \ \&\ \mathrm{III} \longrightarrow T_v^- = A_1 \\ \cdot\ I_3 \ \&\ \mathrm{IV} \longrightarrow T_v^- = A_2 \end{cases}$

◎ Existence of ~~all~~ every types $\{T, L, M\}$

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$\exists$ "$\mathbb{Q}$-split" example for each type (except one)

i.e.

- $E/\mathbb{Q}(t)$ (not just $/\mathbb{C}(t)$ or $k(t)$)
- $\underset{\text{MW gr}}{E(\mathbb{Q}(t))} = E(\bar{\mathbb{Q}}(t)) = E(\mathbb{C}(t))$
- all irred. components of reducible fibres are defined over $\mathbb{Q}$

(in resolving the ADE-singular points on the Weierstrass model, all exceptional curves are defined $/\mathbb{Q}$.)

$\Uparrow$

◎ • construction of "excellent" family for those type $\{T, L, M\}$ where L = root lattice (~30 cases/74)

• idea of "vanishing roots" ($\Leftrightarrow$ "vanishing cycles" in sing. theory)

Ex, 1) $T = E_6$. $L = A_2$. $M = A_2^*$ $\leftarrow$ univ. deform. of of A_2-sing

2) $T = \{0\}$. $L = E_8$. $M = E_8$ of E_8-sing.

E_λ : $y^2 = x^3 + (p_0 + p_1 t + \cdots + p_3 t^3)\, x$

S_λ : $\qquad + (q_0 + q_1 t + \cdots + q_3 t^3 + t^5)$

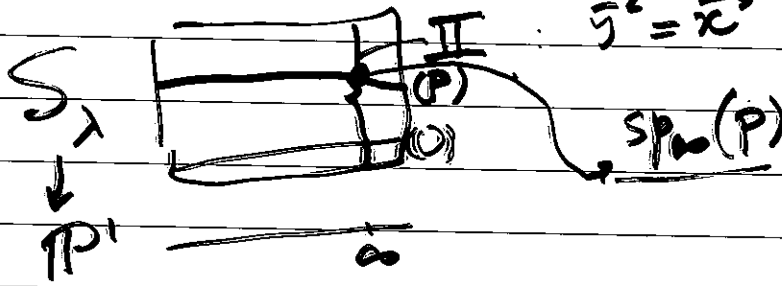
$$\lambda = (p_0, \cdots, p_3, q_0 \cdots q_3)$$

Assume λ generic $/\mathbb{Q}$ $\Rightarrow$ MWL

$(p_i, q_j$ alg. indep. $/\mathbb{Q})$ $\quad E_\lambda(k(t)) \cong E_8$ root lattice

let $k_0 = \mathbb{Q}(\lambda) = \mathbb{Q}(p_i, q_j)$. $\quad P \in$; 240 roots

$k = \overline{k_0}$ alg. closure.



$\bar{y}^2 = \bar{x}^3$ at $t = \infty$ $(\bar{t} = 0)$

$sp_\infty(P) = u \leftrightarrow (P) \cap \varphi^{-1}(\infty)$

$\in \Gamma - \{\text{cusp}\}$. $\quad G_a \underset{\text{gr. hom}}{\cong} \Gamma$

$u \leftrightarrow \left(\frac{1}{u^2}, \frac{1}{u^3}\right)$

$\{P_1, \cdots, P_8\}$ Dynkin basis

$\downarrow \qquad \downarrow$

$u_1, \cdots u_8$

spl. field $\mathbb{Q}(u_1 \cdots u_8) = K_\lambda$

| Galois $W(E_8)$

$\mathbb{Q}(p_0 \cdots q_3) = k_0$

$\Rightarrow \mathbb{Q}[u_1, \cdots u_8] \circlearrowleft W(E_8)$

| integral

$\mathbb{Q}[\lambda] = \mathbb{Q}[\underbrace{p_0, \cdots q_3}_{\text{fund. inv. of } W(E_8)}] = \mathbb{Q}[u_i]^{W(E_8)}$

§ Arith. appl.

(a) $\lambda \to \lambda^0 \in \mathbb{Q}^8$ generic specialize $\Rightarrow$ K | $W(E_8)$-ext. $\mathbb{Q}$ (explicit Ex. PJA 2009)

(b) $u = (u_i) \to (u_i^0) \in \mathbb{Q}^8$, $= u^0$ .. $\Rightarrow E/\mathbb{Q}(t)$ with $\mathrm{rk}\, E(\mathbb{Q}(t)) = 8$ + $\{P_i|_{u^0}\}$ basis.

Remark. Existence above
(or $\mathbb{Q}$-split examples)
$\Longrightarrow$ examples of
the factorization of Δ = discriminant.

Ex. $E: y^2 + 72xy + 10t^2 y$
$= x^3 + 60tx^2 - 15t^2 x + t^6$ $\quad(*)$

$\Delta(t) = \text{const.}\, t^5 (t+32)^3 (t+27)^2$

sing. fibre: I_5, I_3, I_2 II
at $t = 0$, -32, -27, ∞

$T = A_4 \oplus A_2 \oplus A_1$ $\quad$ [OS, No56]

$M = E(\mathbb{Q}(t)) \overset{\exists}{\ni} Q = (t^2 + 24t,\ t^3 + 28t^2)$

Q has height $\frac{1}{30}$, $M = \langle \frac{1}{30} \rangle$
$r = 1$

Fact. $\operatorname{Min} \langle P, P \rangle = \frac{1}{30}$
RES.
$P \neq$ torsion

Rem. $(*)$. affine surface in (x, y, t)-space
has A_4-sing. A_2-sing. A_1-sing
at $(0,0,0)$, $(\cdot, \cdot, -32)$, $(\cdot, \cdot, -27) \in \mathbb{Q}^3$.

Generators of Mordell-Weil group

Suppose $E/_{k(t)}$ given by (min.) Weierstraß form

$$y^2 = x^3 + p(t)x + q(t)$$

$p, q \in k[t]$ (char $\neq 2$)

then

S : rational ell. surface

$$\Longleftrightarrow \begin{cases} \deg p(t) \leq 4, \ \deg q(t) \leq 6 \\ \Delta = 4p(t)^3 + 27q(t)^2 \notin k \end{cases}$$

Generator Theorem

$E/K = k(t)$ s.t. S = rat. ell. surface

$\Rightarrow$ $E(K)$ is generated by rational points of the form

$$P = (x, y), \begin{cases} x = gt^2 + at + b \\ y = ht^3 + ct^2 + dt + e \end{cases}$$

"integral points" (sections) $(g, h, a, \ldots, e \in k)$.

There are at most $\underline{240}$ such P's.

$$240 = \tau(E_8)$$

Rem. 1) "effective"

2) more precise form when $E/_K$ specified. (cf. later)

K3 surfaces $/_{\mathbb{C}}$ X

algebraic cycles on them.

ρ: Picard number $= \mathrm{rk}\, NS(X)$

r: MW rank $\begin{array}{c} X \\ \pi \downarrow \\ \mathbb{P}^1 \end{array}$ elliptic fibration with a section

$$\rho = r + 2 + \sum_{v \in R} (m_v - 1)$$

$R = \{ v \mid \pi^{-1}(v) \text{ is reducible} \}$

$m_v = \#$ irred. comp

$\rho \in \{1, 2, \ldots, 20\}$. $20 = h^{1,1}$

$(\ldots, 22 = b_2$. in ch. $p)$

$r \in \{0, 1, \ldots, 18\}$ $/_{\mathbb{C}}$

$(\ldots, 20$ in ch. $p)$

- Construction of K3 with "big" ρ, ell K3 | r.

start from Kummer surface

$X = Km(A) = \widetilde{(A/\iota_A)}$ min. resol. of A/ι_A 16 node

$$\rho(X) = \rho(A) + 16$$

let $A = C_1 \times C_2$, ell. curve $C_i : y_i^2 = x_i^3 + \cdots$

$$\rho(X) = 18 + h \quad , \quad h = \mathrm{rk}\, \mathrm{Hom}(C_1, C_2)$$
$$\in \{0, 1, 2\} / \mathbb{C}$$

($\because$ 4 in ch. p)

- ell. fibration

$$\begin{array}{ccc} X & K3. & \\ \pi \downarrow & & \pi \longleftrightarrow u \in k(X) \text{ functionfield of } X \\ \mathbb{P}^1 & & \text{"elliptic parameter"} \end{array}$$

Question: Given X

1) find all elliptic parameters $\in k(X)$ (upto equiv.)

2) for u : ell. par, find the equation of $E / k(u)$ (The generic fibre of $X \xrightarrow{\pi} \mathbb{P}^1$, $\pi \longleftrightarrow u$)

- if $X = \overset{Km}{(C_1 \times C_2)}$, $C_1 \not\sim C_2$ non-isog. Question solved. Kuwata-Shioda (Adv.St.P.M. 2008.)

cf. Oguiso's geometric classif into 11 types

Ex. of ell. parameters for $X = \mathrm{Km}(\underbrace{C_1 \times C_2}_{A})$

$$C_1 : y_1^2 = x_1^3 + \cdots$$
$$C_2 : y_2^2 = x_2^3 + \cdots$$

1)
$$\begin{tikzcd} X \arrow[r] \arrow[d, "\pi_1"] & A/\iota_A \arrow[d, "\pi_1"] & A \arrow[l] \arrow[d, "\tilde{\pi}_1 = \mathrm{proj.}"] \\ \mathbb{P}^1 \arrow[r, equal] & C_1/\iota_1 & C_1 \arrow[l] \\ x_1 \arrow[r, equal] & x_1 & (x_1, y_1) \arrow[l] \end{tikzcd}$$

1st Kummer pencil.

x_1 is the ell. par. for π_1

1') x_2 " for π_2 : 2nd Kum. pencil.

2) Note $k(X) = k(x_1, x_2, t)$, $t = \frac{y_2}{y_1}$
(inv. under ι_A)

3) t is an ell. parameter.
$\Longleftrightarrow$ "Inose fibration" on $X = \mathrm{Km}(A)$

($\Rightarrow$ double quotient construction
of so-called Inose-Shioda str.:

which is previously constructed as double cover.

$$\begin{tikzcd} \mathrm{K3}: Y \arrow[dr, "2:1"'] & & A \arrow[dl, "2:1"] \\ & X = \mathrm{Km} & \end{tikzcd}$$

with $\mathrm{Trans}(Y) \cong \mathrm{Trans}(A)$ as Hodge str.

Pf: $(t) = (\mathrm{IV}^*) - (\mathrm{IV}^*)$
at $t=0$ at $t=0$

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$(x_1) = (I_0^*) - (I_0^*)$
$x_1 = 0$ $x_1 = \infty$

$(x_1 - 1) = \cdots$

$(x_1 - \lambda_1) = \cdot$

by 24 (-2) curves A_{ij}, F_i, G_j

$y_1^2 = x_1(x_1-1)(x_1-\lambda_1)$ Legendre.

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eq. of $E/k(t)$

$F^{(2)}_{\alpha\beta}$: $y^2 = x^3 - 3\alpha x + \left(t^2 + \frac{1}{t^2} - 2\beta\right)$

$$\begin{cases} \alpha^3 = j_1 j_2 \\ \beta^2 = (1-j_1)(1-j_2) \end{cases}$$

$F^{(n)}_{\alpha\beta}$: $\cdots + \left(t^n + \frac{1}{t^n} - 2\beta\right)$

K3 if $n = 1, 2, \ldots 6$. Kuwata

MWrk $F^{(n)}_{\alpha\beta}$ computed.

rank formula

$$r^{(n)}_{\alpha\beta} = h + \begin{cases} 4(n-1) & (n \le 5) \\ 16 & (n=6) \end{cases} - \begin{cases} 0 & (j_1 \ne j_2) \\ n & (j_1 = j_2 \ne 0,1) \\ 2n & (j_1 = j_2 = 0 \text{ or } 1) \end{cases}$$

$$h = \mathrm{rk}\,\mathrm{Hom}(C_1, C_2)$$

(N.B. This holds in ch $p > 5$ too.)

e.g.

- if $C_1 \not\sim C_2 \Rightarrow r^{(n)}_{\alpha\beta} = \begin{cases} 4(n-1) & (n \le 5) \\ 16 & (n=6) \end{cases}$

- if $\left.\begin{matrix} C_1 \not\cong C_2 \\ C_1 \sim C_2 \text{ CM.} \\ n = 5 \text{ or } 6 \end{matrix}\right) \Rightarrow \underset{\max}{r^{(n)}_{\alpha\beta} = 18}$

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copy:
page 183 of Kuwata-Shioda

Advanced Studies in Pure Mathematics 50, 2008
Algebraic Geometry in East Asia — Hanoi 2005
pp. 177–215

Elliptic parameters and defining equations for elliptic fibrations on a Kummer surface

Masato Kuwata and Tetsuji Shioda

Type	Singular fibers	MWL	Elliptic parameter u
$\mathcal{J}_1$	$2\mathrm{I}_8 + 8\mathrm{I}_1$	$\mathbf{Z}^2 \oplus \mathbf{Z}/2\mathbf{Z}$	$\frac{tx_1}{x_2}$
$\mathcal{J}_2$	$\mathrm{I}_4 + \mathrm{I}_{12} + 8\mathrm{I}_1$	$A_2^*[2] \oplus \mathbf{Z}/2\mathbf{Z}$	$\frac{t(x_1-\lambda_1)(x_1-x_2)}{x_2(x_2-1)}$
$\mathcal{J}_3$	$2\mathrm{IV}^* + 8\mathrm{I}_1$	$(A_2^*[2])^2$	t
$\mathcal{J}_4$	$4\mathrm{I}_0^*$	$(\mathbf{Z}/2\mathbf{Z})^2$	x_i
$\mathcal{J}_5$	$\mathrm{I}_6^* + 6\mathrm{I}_2$	$(\mathbf{Z}/2\mathbf{Z})^2$	$\frac{(x_1-x_2)(\lambda_2(x_1-\lambda_1)+(\lambda_1-1)x_2)}{(\lambda_2 x_1-x_2)(x_1-\lambda_1+(\lambda_1-1)x_2)}$
$\mathcal{J}_6$	$2\,\mathrm{I}_2^* + 4\mathrm{I}_2$	$(\mathbf{Z}/2\mathbf{Z})^2$	$\frac{x_1}{x_2}$
$\mathcal{J}_7$	$\mathrm{I}_4^* + 2\mathrm{I}_0^* + 2\mathrm{I}_1$	$\mathbf{Z}/2\mathbf{Z}$	$\frac{(x_2-\lambda_2)(x_1-x_2)}{(x_2-1)(\lambda_2 x_1-x_2)}$
$\mathcal{J}_8$	$\mathrm{III}^* + \mathrm{I}_2^* + 3\mathrm{I}_2 + \mathrm{I}_1$	$\mathbf{Z}/2\mathbf{Z}$	$-\frac{(x_2-\lambda_2)(x_1-x_2)}{\lambda_2(\lambda_2-1)x_1(x_1-1)}$
$\mathcal{J}_9$	$\mathrm{II}^* + 2\mathrm{I}_0^* + 2\mathrm{I}_1$	$\{0\}$	$\frac{(x_2-\lambda_2)(x_1-x_2)(\lambda_2 x_1(x_1-1)+(\lambda_1-1)(x_2-1)(\lambda_2 x_1-x_2))}{(x_2-1)(\lambda_2 x_1-x_2)(\lambda_2 x_1(x_1-1)+(\lambda_1-1)(x_2-\lambda_2)(x_1-x_2))}$
$\mathcal{J}_{10}$	$\mathrm{I}_8^* + \mathrm{I}_0^* + 4\mathrm{I}_1$	$\{0\}$	$\frac{(x_2-\lambda_2)(x_1-x_2)((\lambda_1-1)(x_2-1)(\lambda_2 x_1-x_2)+\lambda_2 x_1(x_1-1))}{x_2(x_2-1)(x_1-1)(\lambda_2 x_1-x_2)}$
$\mathcal{J}_{11}$	$2\mathrm{I}_4^* + 4\mathrm{I}_1$	$\{0\}$	$\frac{x_2(x_2-\lambda_2)(x_1-x_2)}{x_1(x_2-1)(\lambda_2 x_1-x_2)}$

Table 1. Results

Notation

X : complex K3 surface, (n.s. projective) $k = \mathbb{C}$.

$\boxed{NS(X)}$: <u>Néron-Severi</u> group / lattice

$\perp$ $\overset{\text{prim.}}{\subset}$ $H^2(X, \mathbb{Z}) \simeq U^{\oplus 3} \oplus E_8[-1]^{\oplus 2}$.

$\cup$ primitive $\quad U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

$\boxed{T(X)}$: <u>lattice of transcendental cycles on X.</u>

$\lambda(X) = rk\, T(X) = 22 - \rho(X)$

Lefschetz number, $\quad$ ↑ Picard number

X elliptic K3 surface

$$O \uparrow \downarrow f$$

$\mathbb{P}^1$

$X = (X, f)$, with $\underset{\text{zero}}{\text{section}}$ (O)

$\boxed{MW(X)} = \{\text{sections of } f\}$

$\underset{\text{identify}}{=} \{k(t)\text{-rational points of } E/k(t) = \text{generic fibre}\}$

<u>Mordell-Weil lattice</u>,

$\langle\ ,\ \rangle$: height pairing on $MW(X)$

new 2007.

cf old: A note on KS & SP. Proc. Japan Acad. (2000) (4)

§. Main Results :

$\lambda = 4 - h$ (rk 24)

Th 1. $$\boxed{T(F^{(n)}_{\alpha_\beta}) \underset{\text{lattice}}{\simeq} T(F^{(1)}_{\alpha\beta})[n]}$$

$\forall \alpha, \beta, \quad \forall n \le 6$

Cor $$\det T(F^{(n)}_{\alpha_\beta}) = \det T(F^{(1)}_{\alpha_\beta}) \cdot n^{\lambda}$$

$\lambda = 4 - h$

Th 2 $$\boxed{\det NS(F^{(n)}_{\alpha_\beta}) = \det \mathrm{Hom}(C_1, C_2) \cdot n^{\lambda}}$$

rkNS ρ''

$\mathrm{Hom}(C_1, C_2) \ni \varphi$

integral lattice with norm $(\varphi) = 2\deg(\varphi)$

$\mathrm{rk}\, MW = r^{(n)}$

Th 3 $$\boxed{\det MW(F^{(n)}_{\alpha_\beta}) = \frac{\det \mathrm{Hom}(C_1, C_2) \cdot n^{\lambda}}{i^{n} \cdot c(n)}}, \quad c(n) = \begin{cases} 1 & (n = 1, 5, 6) \\ 3^2 & (n = 2, 4) \\ 4^2 & (n = 3) \end{cases}$$

where $(\alpha, \beta) \longleftrightarrow \{j_1, j_2\}$ with

$$i = \begin{cases} 1 & \text{if } j_1 \neq j_2 \\ 2 & \text{if } j_1 = j_2, \neq 0 \text{ or } 1 \\ 3 & \text{if } j_1 = j_2 = 0 \\ 4 & \text{if } j_1 = j_2 = 1. \end{cases}$$

[call Case (i) later.]

Th 3 (continued)

$MW(E^n_\beta)$ is torsion-free except for the following case;

$$\begin{cases} \text{Case (3)} & n = 2, 4 \text{ or } 6, \\ \text{Case (4)} & n = 3, 6. \end{cases}$$

In these cases, the RHS should be multiplied by

$$|MW_{tor}|^2 \text{ with } MW_{tor} \simeq \begin{cases} \mathbb{Z}/3 \\ (\mathbb{Z}/2)^2 \end{cases}.$$

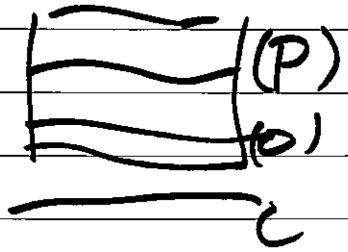
Th 4. Assume $h = 0$, i.e. $C_1 \not\sim C_2$ not-isogenous

Then MWL $\boxed{MW(F^{(n)}_{\alpha\beta}) = M^{(n)}_{gen}}$ has the following invariants: ⟨ is indep. of α, β, and

n	rank	det	μ min.norm	δ center density
1	0	1	—	
2	4	$2^4/3^2$	$4/3$	
3	8	$3^4/4^2$	2	
4	12	$4^4/3^2$	$8/3$	
5	16	5^4	4	$1/5^2$
6	16	6^4	4	$1/6^2$

~~✗~~ Integral sections (no time)

everywhere



$P \in M = \{\text{sections}\}$

P is "everywhere integral"

$\overset{}{\underset{\text{def}}{\Longleftrightarrow}} \ (P) \cap (0) = \phi$

· Th: $\mathcal{P} = \{ P \mid \text{(ev.) integral sections} \} \subset M$ is a finite set

☺ $\Leftarrow$ ~~height formula~~

· Assume $C = \mathbb{P}^1$.

$\underset{E}{S}$: $y^2 = x^3 + A(t)x + B(t)$. $A, B \in \mathbb{C}[t]$ ($k[t]$)

min W-eq. $\chi = 1$ (RES)
$\chi = 2$ (K3)

$$\mathcal{P} \ni P : \begin{cases} x(t) = x_0 + x_1 t + \cdots + x_{2\chi} t^{2\chi} \\ y(t) = y_0 + \cdots + y_{3\chi} t^{3\chi} \end{cases}$$

$$P \mapsto z(P) = (x_i, y_j) \in \mathbb{A}^{5\chi+2}$$

"coord" of P affine sp.

$I \subset R = k[x_i, y_j]$ pol. ring

"defining eq of $\mathcal{P}$

· $\dim_k (R/I) = \sum_{P \in \mathcal{P}} \text{mult}(P)$

$n = |\mathcal{P}|$

$I = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_n$ primary decom

$\mathfrak{q}_i \rightsquigarrow \sqrt{\mathfrak{q}_i} = \mathfrak{p}_i$ prime ideal $\underset{\text{max. ideal}}{\shortparallel} \leftrightarrow Z(P)$

$\mathfrak{q}_i \longleftrightarrow P_i$ for $P \in \mathcal{P}$

Question $\begin{cases} \text{(i) } n = |\mathcal{P}| \\ \text{(ii) } \mathrm{mult}(P) \ ? \\ \text{(iii) } \dim_k R/I. \end{cases}$

Complete Answer for $\chi = 1$, RES

Th. $\begin{cases} \text{(i) } n \leq 240 \\ \text{(ii) } \mathrm{mult}(P) = \#\text{"dirty-valued roots" in the "root graph" assoc. with } P \ (P \in \mathcal{P}) \\ \text{(iii) } \dim_k R/I = 240 - \nu(T) \\ \quad \left(\begin{array}{l} T = \oplus T_v, \ \text{ADE} \\ \nu(T) = \#\text{roots in } T \end{array} \right). \end{cases}$

- lots of examples ⇐
- Grühn Tain

Question: ? for $\chi = 2$

[Case $\underline{\chi = 1}$ RES. 240 roots in the E_8-frame is "loc. const". preservation of 240 roots!]

no such analog for K3 !?

Reference : see

"Groebner basis, MWL and deformation of singularities, I, II".

to appear soon: Jan. issue of Proc. Japan Academy (2010)