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F(R) SUPERGRAVITY and EARLY UNIVERSE

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- Motivation: Big Bang, Inflation and High Energy Physics
- Basics of cosmological inflation
- Starobinsky model of chaotic inflation
- New: F(R) supergravity ($N = 1$ in 4D)
- New: AdS bound in F(R) supergravity
- New: Embedding of Starobinsky model into F(R) supergravity
- Conclusion and Outlook

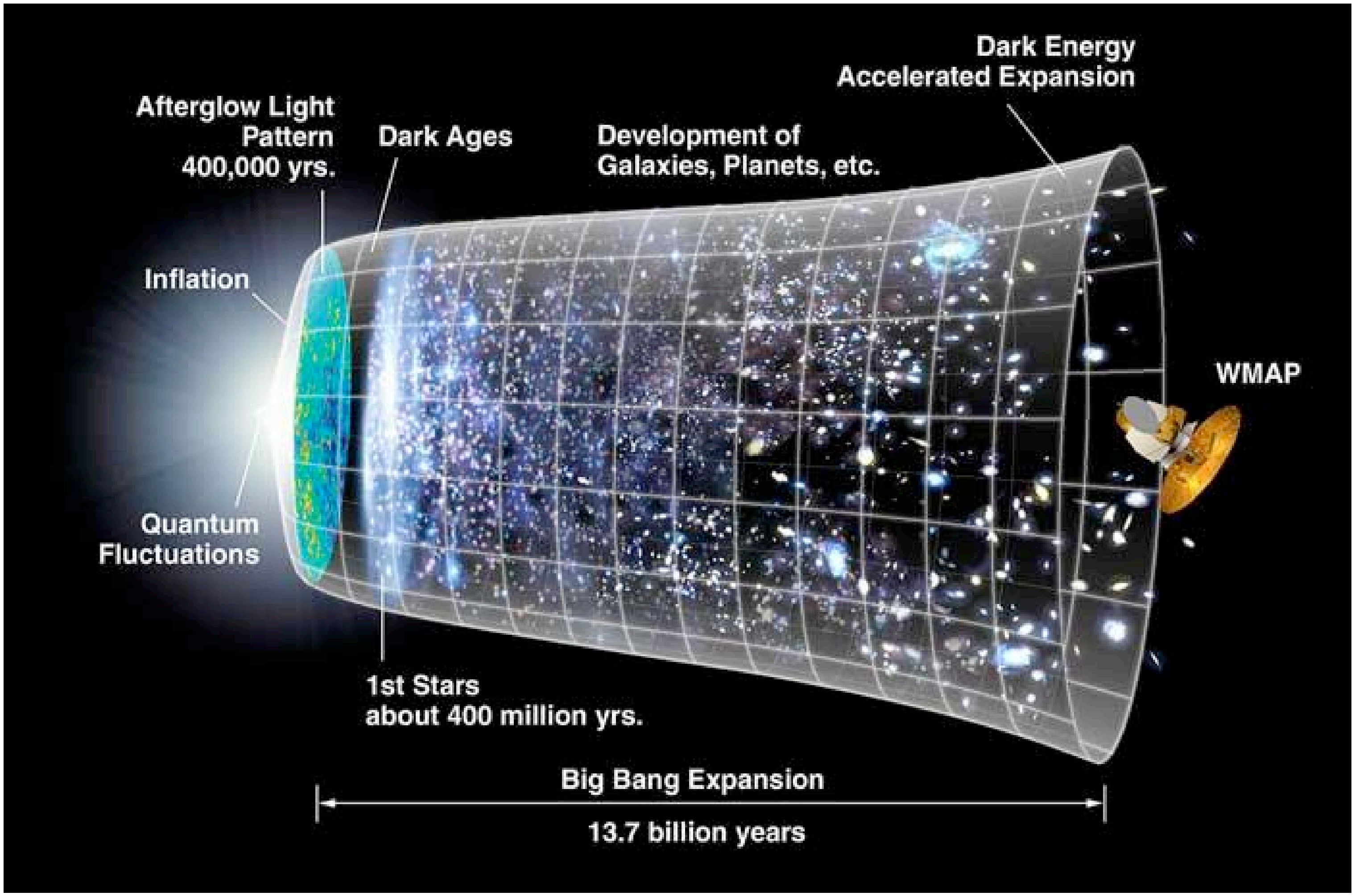
Collaborators

- Prof. Jim Gates Jr. (Univ. of Maryland at College Park, USA)
- Prof. A. Starobinsky (Landau Inst. for Theor. Physics, Russia)
- Dr. Nico Yunes (Princeton Univ., USA)
- Masao Iihoshi, Natsuke Watanabe and Sho Kaneda (my students at Tokyo Metropolitan University, Japan)

Our References

- Adv. High Energy Phys. 521389 (2008), with M. Iihoshi
- Phys. Lett. B684 (2009) 207 and B692 (2010) 272
- Classical and Quantum Gravity 26 (2009) 135006 and 27 (2010) 145016
- Phys. Rev. D80 (2009) 065003, with S.J. Gates Jr. and N. Yunes
- “Invisible Universe”, Paris, 2009; AIP. Conf. Proc. 1241 (2010) 609; 613
- Mod. Phys. Lett. A25 (2010) 2753, with S. Kaneda and N. Watanabe
- arXiv:1011.nnnn (PRL), with A. Starobinsky (to appear Nov 2nd)

History of our Universe



Inflation in Early Universe

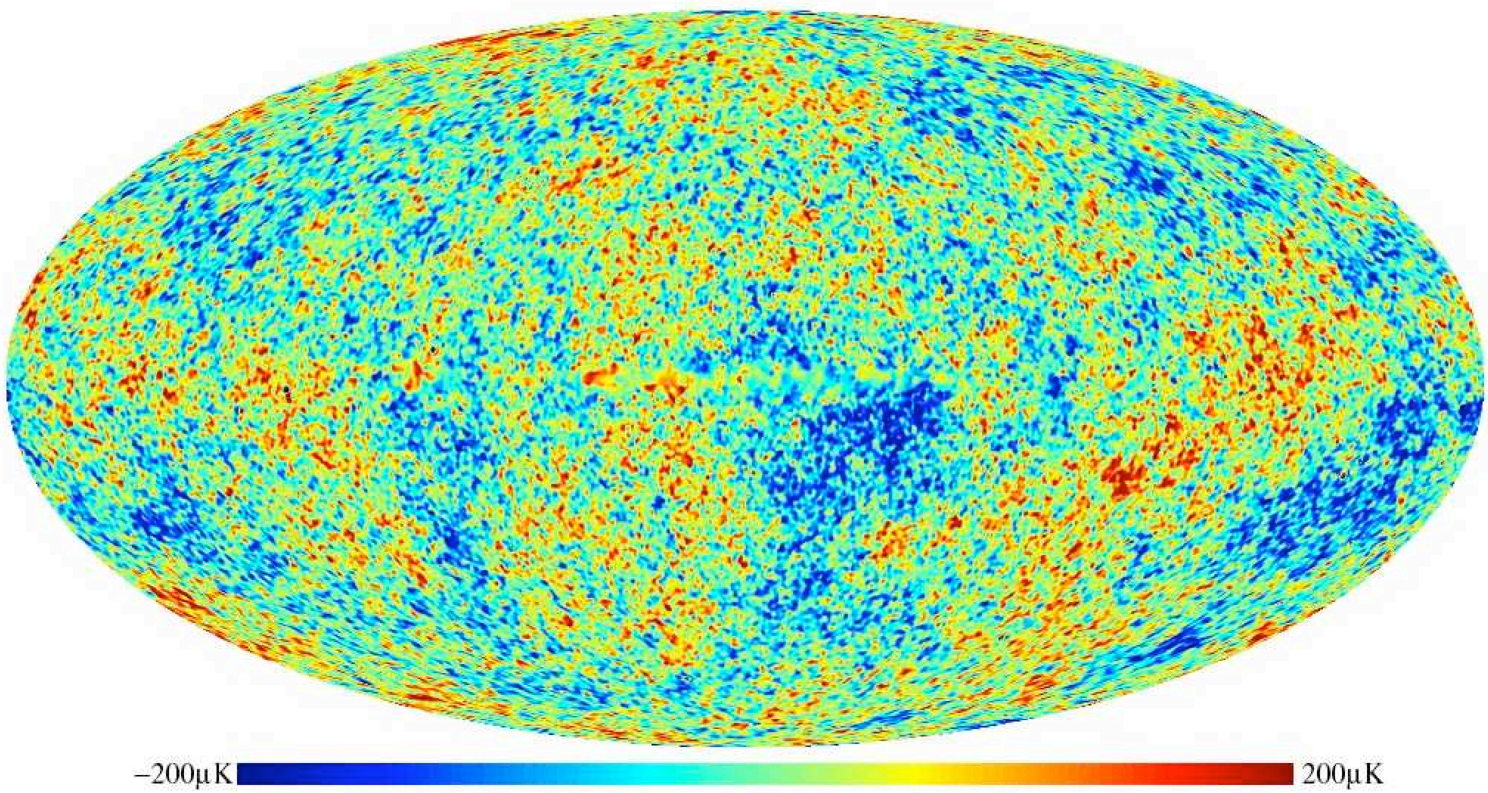
- Cosmological inflation (a phase of 'rapid' accelerated expansion) predicts homogeneity of our Universe at large scales, its spatial flatness, large size and entropy, and the almost scale-invariant spectrum of cosmological perturbations (in agreement with the WMAP measurements of the CMB radiation spectrum)

- Inflation is a paradigm, not a theory! Known theoretical mechanisms of inflation use a slow-roll scalar field (called inflaton) with proper scalar potential

- The scale of inflation is well beyond the electro-weak scale (near/below the Grand Unification scale) – it is High-Energy Physics beyond the SM !

- The nature of the inflaton and the origin of its scalar potential are the big mysteries. Einstein gravity alone is not enough.

CMB radiation (Mona Lisa of Cosmology)



FLRW metric and cosmological acceleration (Dark Energy)

- The main Cosmological Principle of a spatially homogeneous and isotropic $(1+3)$-dimensional universe (at large scales) gives rise to the FLRW metric

$$ds^2_{\mathsf{FLRW}} = dt^2 - a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2 d\Omega^2\right]$$

where the function $a(t)$ is known as the scale factor in 'cosmic' (co-moving) co-ordinates (t, r, θ, ϕ), and k is the FLRW topology index, $k = (-1, 0, +1)$. The FLRW metric (1) admits a 6-dimensional isometry group G that is either $SO(1,3)$, $E(3)$ or $SO(4)$, acting on the orbits $G/SO(3)$, with the spatial 3-dimensional sections H^3, E^3 or S^3, respectively. Important notice: Weyl tensor $C^{\mathsf{FLRW}}_{ijkl} = 0$.

- The early Universe inflation (acceleration) means

$$\ddot{a}(t) > 0 \text{ , or equivalently , } \frac{d}{dt}\left(\frac{H^{-1}}{a}\right) < 0$$

where $H = \dot{a}/a$ is Hubble function. The amount of inflation (# e-foldings) is given by

$$N_e = \ln \frac{a(t_{\mathsf{end}})}{a(t_{\mathsf{start}})} = \int_{t_{\mathsf{start}}}^{t_{\mathsf{end}}} H\, dt \approx \frac{1}{M^2_{\mathsf{Pl}}} \int_{\phi_{\mathsf{end}}}^{\phi} \frac{V}{V'}\, d\phi$$

Our preferences (part of our motivation and working philosophy)

- Going beyond Einstein is a must, both from the experimental viewpoint (eg., due to the existence of DE) and from the theoretical viewpoint (eg., due to the UV incompleteness of Einstein Gravity, and the need of quantum unification of Gravity and SM)

- DE is unclustered, seen by gravitational interaction only. Like Einstein gravity, the origin of DE should be geometrical, ie. be closely related to space-time and gravity. We opt for introducing the higher-order curvature terms on the l.h.s. of Einstein equations, and extending gravity to supergravity. Both are required by Superstrings too.

- I am going to talk about inflation (primordial DE) in gravity and supergravity. It is the scalar curvature dependence of the gravitational effective action that is (arguably) most relevant to the large-scale dynamics $a(t)$.

Simple example: Starobinsky model (1980)

In 4 dimensions, all the independent quadratic curvature invariants are $R^{\mu\nu\lambda\rho}R_{\mu\nu\lambda\rho}$, $R^{\mu\nu}R_{\mu\nu}$ and R^2. However,

$$\int d^4x\,\sqrt{-g}\left(R^{\mu\nu\lambda\rho}R_{\mu\nu\lambda\rho} - 4R^{\mu\nu}R_{\mu\nu} + R^2\right)$$

is topological for any metric, while

$$\int d^4x\,\sqrt{-g}\left(3R^{\mu\nu}R_{\mu\nu} - R^2\right)$$

is also topological for any FLRW metric. Hence, the FLRW-relevant quadratically-corrected gravity action is $(8\pi G_N = 1)$

$$S = -\frac{1}{2}\int d^4x\,\sqrt{-g}\left(R - \alpha R^2\right)$$

known as the Starobinsky model. It has a stable inflationary solution (attractor!). In particular, for $H \gg M$, one finds

$$H \approx \left(\frac{M}{6}\right)^2 (t_{\text{end}} - t)$$

It is the realization of chaotic inflation (chaotic = chaotic initial conditions).

$f(R)$-gravity equations in (flat) FLRW Universe

The Starobinsky model is the special case of the $f(R)$-gravity models

$$S_f = -\frac{1}{16\pi G_N}\int d^4x\,\sqrt{-g}\,f(R)$$

In the absence of matter, the gravitational (trace) equation of motion is of the 4th-order with respect to the time derivative,

$$\frac{3}{a^3}\frac{d}{dt}\left(a^3\frac{df'(R)}{dt}\right) + Rf'(R) - 2f(R) = 0$$

where we have used $H = \frac{\dot{a}}{a}$ and $R = -6(\dot{H} + 2H^2)$. The static de-Sitter solutions correspond to the roots of the equation $Rf'(R) = 2f(R)$.

The 00-component of the gravitational equaqtions is of the 3rd order:

$$3H\frac{df'(R)}{dt} - 3(\dot{H} + H^2)f'(R) - \frac{1}{2}f(R) = 0$$

The (classical and quantum) stability conditions are $f'(R) > 0$ and $f''(R) < 0$, resp.

$f(R)$-gravity = scalar-tensor gravity

Any $f(R)$ gravity is (classically) equivalent to the scalar-tensor gravity having the (extra) propagating scalar field ϕ (Whitt, Maeda).

The equivalence is established via a Legendre-Weyl transform. First, the $f(R)$-gravity action can be rewritten to

$$S_A = \frac{-1}{2\kappa^2} \int d^4x \sqrt{-g} \, \{AR - Z(A)\}$$

where the real scalar $A(x)$ is related to the scalar curvature R by the Legendre transformation,

$$R = Z'(A) \qquad \text{and} \qquad f(R) = RA(R) - Z(A(R))$$

and $\kappa^2 = 8\pi G_N = M_{\rm Pl}^{-2}$.

Second, a Weyl transformation of the metric

$$g_{\mu\nu}(x) \to \exp\left[\frac{2\kappa\phi(x)}{\sqrt{6}}\right] g_{\mu\nu}(x)$$

with the arbitrary field parameter $\phi(x)$ yields

$$\sqrt{-g}\, R \to \sqrt{-g}\ \exp\left[\frac{2\kappa\phi(x)}{\sqrt{6}}\right]\left\{R - \sqrt{\frac{6}{-g}}\partial_\mu\left(\sqrt{-g}g^{\mu\nu}\partial_\nu\phi\right)\kappa - \kappa^2 g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\right\}$$

Hence, when choosing

$$A(\kappa\phi) = \exp\left[\frac{-2\kappa\phi(x)}{\sqrt{6}}\right]$$

and ignoring a total derivative, we can rewrite the action to the form

$$S[g_{\mu\nu},\phi] = \int d^4x\,\sqrt{-g}\left\{\frac{-R}{2\kappa^2} + \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + \frac{1}{2\kappa^2}\exp\left[\frac{4\kappa\phi(x)}{\sqrt{6}}\right] Z(A(\kappa\phi))\right\}$$

in terms of the physical (and canonically normalized) scalar field $\phi(x)$ (no higher derivatives and no ghosts). As a result, we arrive at the standard action of the real dynamical scalar field $\phi(x)$ minimally coupled to Einstein gravity and having the scalar potential

$$V(\phi) = -\frac{M_{\mathrm{Pl}}^2}{2}\exp\left\{\frac{4\phi}{M_{\mathrm{Pl}}\sqrt{6}}\right\} Z\left(\exp\left[\frac{-2\phi}{M_{\mathrm{Pl}}\sqrt{6}}\right]\right)$$

Dual scalar potential in the Starobinsky model

In the context of inflation, the scalaron ϕ is identified with inflaton. This inflaton has clear origin as the conformal mode of a metric (dilaton).

In the case of $f(R) = R - R^2/M^2$, the inflaton scalar potential reads

$$V(y) = V_0 \left(e^{-y} - 1\right)^2$$

where we have introduced the notation

$$y = \sqrt{\frac{2}{3}} \frac{\phi}{M_{\mathrm{Pl}}} \qquad \text{and} \qquad V_0 = \frac{1}{8} M_{\mathrm{Pl}}^2 M^2$$

Note the appearance of the inflaton vacuum energy V_0 driving inflation. The end of inflation (graceful exit) also exist in this model.

(Standard) slow-roll inflation

The slow-roll inflation parameters are defined by

$$\varepsilon(\phi) = \frac{1}{2} M_{\mathrm{Pl}}^2 \left(\frac{V'}{V}\right)^2 \qquad \text{and} \qquad \eta(\phi) = M_{\mathrm{Pl}}^2 \frac{V''}{V}$$

The necessary condition for the slow-roll approximation is the smallness of the inflation parameters

$$\varepsilon(\phi) \ll 1 \qquad \text{and} \qquad |\eta(\phi)| \ll 1$$

The first condition implies $\ddot{a}(t) > 0$. The second one guarantees that inflation lasts long enough, via domination of the friction term in the inflaton equation of motion, $3H\dot{\phi} = -V'$.

The primordial spectrum is proportional to k^{n-1}, in terms of the comoving wave number k and the spectral index n. The slope n_s of the scalar power spectrum, associated with density perturbations, in theory is (Liddle, Lyth): $n_s = 1 + 2\eta - 6\varepsilon$, the slope of the tensor primordial spectrum, associated with gravitational waves, is $n_t = -2\varepsilon$, and the scalar-to-tensor ratio is $r = 16\varepsilon$.

Physical observables in Starobinsky model

In Starobinsky model, one finds (Chibisov, Mukhanov)

$$n_s = 1 - \frac{2}{N_e} + \frac{3\ln N_e}{2N_e^2} - \frac{2}{N_e^2} + \mathcal{O}\left(\frac{\ln^2 N_e}{N_e^3}\right)$$

and $r \approx \frac{12}{N_e^2} \approx 0.004$ with $N_e \approx 55$. It is to be compared to

$\boxed{\text{the experimental (WMAP) values } n_s = 0.963 \pm 0.012}$

and $r < 0.24$ (with 95 % confidence).

The amplitude of the initial perturbations, $\Delta_R^2 = M_{\mathrm{Pl}}^4 V/(24\pi^2\varepsilon)$, is another physical observable, whose experimental value is $\left(\frac{V}{\varepsilon}\right)^{1/4} = 0.027\, M_{\mathrm{Pl}} = 6.6 \times 10^{16}$ GeV. It determines the normalization of the R^2-term in the action,

$$\frac{M}{M_{\mathrm{Pl}}} = 4 \cdot \sqrt{\frac{2}{3}} \cdot (2.7)^2 \cdot \frac{e^{-y}}{(1-e^{-y})^2} \cdot 10^{-4} \approx (3.5 \pm 1.2) \cdot 10^{-6}$$

Some more lessons

- The discriminants among inflationary models are the values of n_s and r

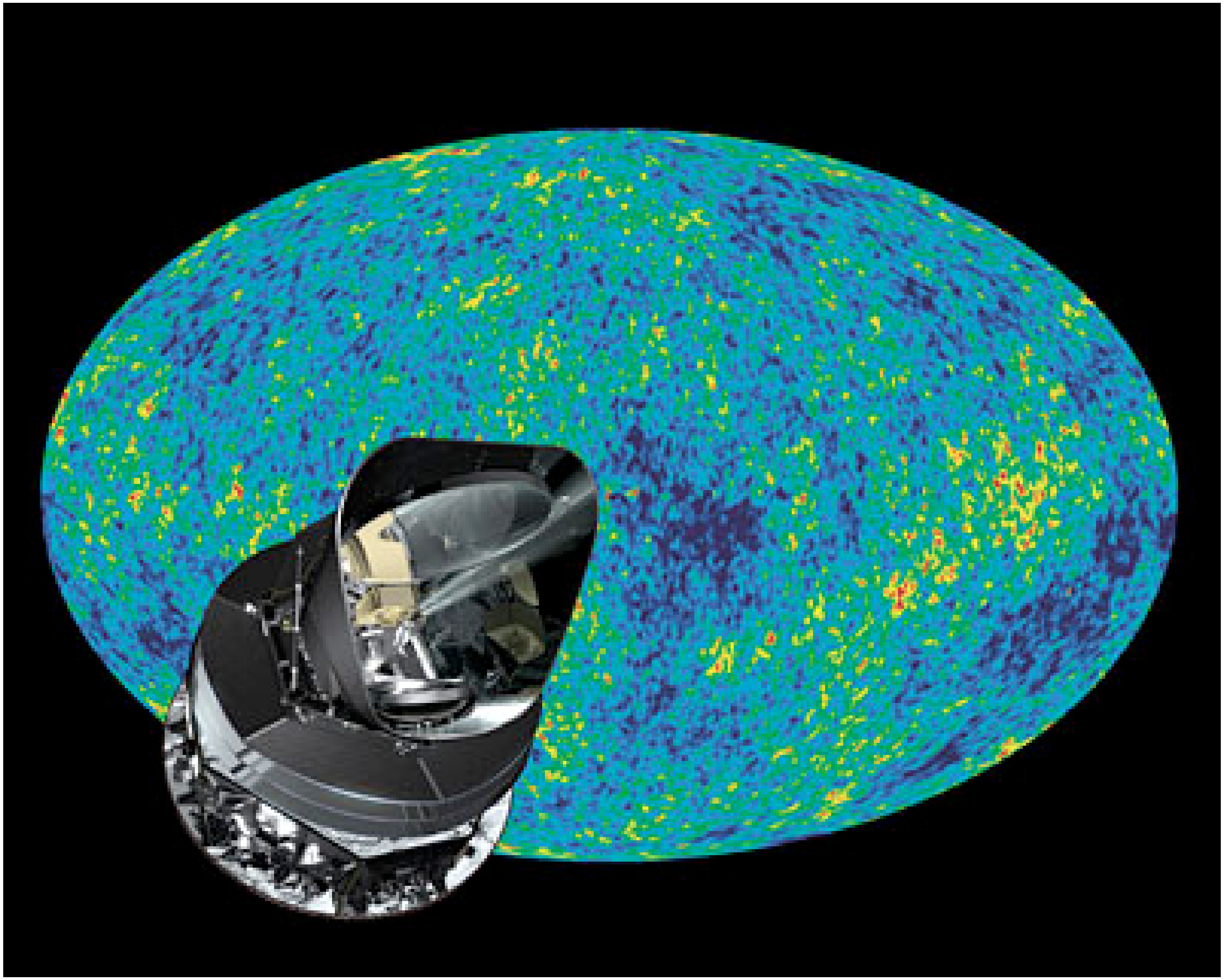
- The Starobinsky model of chaotic inflation (1980) is still viable, and gives the very simple explanation to the WMAP-observed value of n_s. The crucial test is a measurement of r due to primordial gravitational waves

- The scalaron (inflaton) is going to decay by the end of inflation due to one-loop gravitational corrections (Starobinsky)

- All inflationary models of chaotic inflation based on $f(R) = R + \tilde{f}(R)$ gravity are close to the simple Starobinsky model (over some range of R) with $\tilde{f}(R) \approx R^2 A(R)$ and the slowly varying function $A(R)$,

$$\left|A'(R)\right| \ll \frac{A(R)}{R} , \qquad \left|A''(R)\right| \ll \frac{A(R)}{R^2}$$

Planck mission: 0.5% accuracy in CMB expected



Next step: Supergravity

- Supersymmetry is the symmetry between bosons and fermions, it is well motivated in particle physics beyond the SM, and it is needed for consistency of strings. Supergravity is the theory of local supersymmetry. Supergravity is the only known consistent route to couple spin-3/2 particles (gravitinos).
- Most of studies of superstring- and brane- cosmology are based on their effective description in the 4-dimensional $N = 1$ supergravity.
- An $N = 1$ locally supersymmetric extension of $f(R)$ gravity is possible (Gates Jr., SVK, 2009). It is also non-trivial because there should be no ghosts, and the auxiliary freedom (Gates Jr., 1996) is to be preserved. The new supergravity action is classically equivalent to the standard $N = 1$ Poincaré supergravity coupled to a dynamical chiral superfield whose Kähler potential and superpotential are dictated by a single holomorphic function. The inflaton arises as the superconformal mode of a supervielbein.

A possible connection to the Loop Quantum Gravity was investigated by Gates Jr., N. Yunes and SVK, in Phys. Rev. D80 (2009) 065003, arXiv:0906.4978 [hep-th].

Inflation in Supergravity (Review)

- There is a generic problem to realize an (F-term) slow-roll inflation in supergravity (Murayama,Suzuki,Yanagida,Yokoyama,1994). Techically, the problem is due to a presence of the factor $\exp(K/M^2_{\rm Pl})$ in the F-term generated scalar potential of the matter-coupled supergravity. Given the naive (tree level) Kähler potential $K \propto \overline{\Phi}\Phi$, one gets the inflaton scalar potential $V \propto \exp(|\Phi|^2/M^2_{\rm Pl})$ that is too steep to support chaotic inflation (so-called η-problem when $\eta \approx 1$ or, equivalently, $m^2_{\rm inflaton} \approx V_0/M^2_{\rm Pl} \approx H^2$ that is unacceptable and thus requires unnatural fine-tuning)

- Two cures are known in the literature. The 1st one is the D-term mechanism (Binetruy,Dvali,1996), where inflation is generated in the gauge sector and is highly sensitive to the gauge charges. The 2nd proposal is to assume that the Kähler potential K does not depend upon some scalars (= flat directions), and then add a desired scalar potential for flat directions, by identifying one of them with the inflaton (Kawasaki,Yamaguchi,Yanagida,2000). None of those approaches is geometrical since both refer to the matter sector.

Our Proposal for Inflation in Supergravity

- construct a locally $N = 1$ supersymmetric extension of $f(R)$ gravity (we call it $F(R)$-supergravity) in four spacetime dimensions,

- study applications of the $F(R)$ supergravity to HEP and early Universe,

and, in particular,

- find embedding of the Starobinsky model of chaotic inflation into the $F(R)$-supergravity.

Basic facts about 4-dim, $N = 1$ supergravity in superspace

A concise and manifestly supersymmetric description of supergravity is given by Superspace. We use here the natural units $c = \hbar = \kappa = 1$.

The chiral superspace density (in the supersymmetric gauge-fixed form) reads

$$\mathcal{E}(x,\theta) = e(x)\left[1 - 2i\theta\sigma_a\bar{\psi}^a(x) + \theta^2 B(x)\right] \ , \tag{1}$$

where $e = \sqrt{-\det g_{\mu\nu}}$, $g_{\mu\nu}$ is a spacetime metric, $\psi^a_\alpha = e^a_\mu \psi^\mu_\alpha$ is a chiral gravitino, $B = S - iP$ is the complex scalar auxiliary field. We use the lower case middle greek letters $\mu, \nu, \ldots = 0, 1, 2, 3$ for curved spacetime vector indices, the lower case early latin letters $a, b, \ldots = 0, 1, 2, 3$ for flat (target) space vector indices, and the lower case early greek letters $\alpha, \beta, \ldots = 1, 2$ for chiral spinor indices. Supergravity $\neq$ curved Superspace (off-shell SUSY constraints needed)!

The solution of the superspace Bianchi identitiies and the constraints defining the $N = 1$ Poincaré-type *minimal* supergravity results in only three covariant tensor superfields $\mathcal{R}$, $\mathcal{G}_a$ and $\mathcal{W}_{\alpha\beta\gamma}$, subject to the off-shell relations:

$$\mathcal{G}_a = \bar{\mathcal{G}}_a \ , \qquad \mathcal{W}_{\alpha\beta\gamma} = \mathcal{W}_{(\alpha\beta\gamma)} \ , \qquad \bar{\nabla}_{\dot{\alpha}}\mathcal{R} = \bar{\nabla}_{\dot{\alpha}}\mathcal{W}_{\alpha\beta\gamma} = 0 \ , \tag{2}$$

and

$$\bar{\nabla}^{\dot{\alpha}}\mathcal{G}_{\alpha\dot{\alpha}} = \nabla_\alpha\mathcal{R} \ , \qquad \nabla^\gamma\mathcal{W}_{\alpha\beta\gamma} = \tfrac{i}{2}\nabla_\alpha{}^{\dot{\alpha}}\mathcal{G}_{\beta\dot{\alpha}} + \tfrac{i}{2}\nabla_\beta{}^{\dot{\alpha}}\mathcal{G}_{\alpha\dot{\alpha}} \ , \tag{3}$$

where $(\nabla_\alpha, \bar{\nabla}_{\dot{\alpha}}.\nabla_{\alpha\dot{\alpha}})$ represent the curved superspace $N = 1$ supercovariant derivatives, and bars denote complex conjugation.

The covariantly chiral complex scalar superfield $\mathcal{R}$ has the scalar curvature R as the coefficient at its θ^2 term, the real vector superfield $\mathcal{G}_{\alpha\dot{\alpha}}$ has the traceless Ricci tensor, $R_{\mu\nu} + R_{\nu\mu} - \frac{1}{2}g_{\mu\nu}R$, as the coefficient at its $\theta\sigma^a\bar{\theta}$ term, whereas the covariantly chiral, complex, totally symmetric, fermionic superfield $\mathcal{W}_{\alpha\beta\gamma}$ has the self-dual part of the Weyl tensor $C_{\alpha\beta\gamma\delta}$ as the coefficient at its linear θ^δ-dependent term. A generic higher-derivative supergravity Lagrangian (e.g., representing the supergravitational part of the superstring effective action) is given by

$$\mathcal{L} = \mathcal{L}(\mathcal{R}, \mathcal{G}, \mathcal{W}, \ldots) \tag{4}$$

where the dots stand for arbitrary supercovariant derivatives of the superfields.

New proposal: $F(\mathcal{R})$ supergravity

Let's concentrate on the scalar sector of a generic higher-derivative supergravity (4), which is most relevant to cosmology, by ignoring the tensor superfields $\mathcal{W}_{\alpha\beta\gamma}$ and $\mathcal{G}_{\alpha\dot{\alpha}}$, as well as the derivatives of the scalar superfield $\mathcal{R}$,

$$S_F = \int d^4x d^2\theta\, \mathcal{E} F(\mathcal{R}) + \text{H.c.} \tag{5}$$

with a holomorphic function $F(\mathcal{R})$. Besides manifest local $N = 1$ supersymmetry, the action (5) also possess the auxiliary freedom, since the auxiliary field B does not propagate. It distinguishes the action (5) from other possible truncations of eq. (4). The action (5) gives rise to the spacetime torsion generated by gravitino, while its bosonic terms have the form

$$S_f = \int d^4x \sqrt{-g}\, f(R) \tag{6}$$

Hence, eq. (5) can also be considered as the locally $N = 1$ supersymmetric extension of the $f(R)$-type gravity. Supergravity is very restrictive! and has more particles and fields.

Legendre-Weyl-Kähler transformation in supergravity

The superfield action (5) is classically equivalent to

$$S_V = \int d^4x d^2\theta \, \mathcal{E} \left[\mathcal{Z}\mathcal{R} - V(\mathcal{Z}) \right] + \text{H.c.} \tag{7}$$

with the covariantly chiral superfield $\mathcal{Z}$ as the Lagrange multiplier. Varying the action (7) with respect to $\mathcal{Z}$ gives back the original action (5) provided that

$$F(\mathcal{R}) = \mathcal{R}\mathcal{Z}(\mathcal{R}) - V(\mathcal{Z}(\mathcal{R})) \tag{8}$$

where the function $\mathcal{Z}(\mathcal{R})$ is defined by inverting the function

$$\mathcal{R} = V'(\mathcal{Z}) \tag{9}$$

Equations (8) and (9) define the superfield Legendre transform, and imply

$$F'(\mathcal{R}) = Z(\mathcal{R}) \qquad \text{and} \qquad F''(\mathcal{R}) = Z'(\mathcal{R}) = \frac{1}{V''(\mathcal{Z}(\mathcal{R}))} \tag{10}$$

where $V'' = d^2V/d\mathcal{Z}^2$. The second formula (10) is the duality relation between the supergravitational function F and the chiral superpotential V.

A super-Weyl transform of the acton (7) can be done entirely in superspace. In terms of components, the super-Weyl transform amounts to a Weyl transform, a chiral rotation and a (superconformal) S-supersymmetry transformation (Howe). The chiral density superfield $\mathcal{E}$ is a chiral compensator of the super-Weyl transformations,

$$\mathcal{E} \to e^{3\Phi}\mathcal{E} \ , \tag{11}$$

whose parameter Φ is an arbitrary covariantly chiral superfield, $\bar{\nabla}_{\dot{\alpha}}\Phi = 0$. Under the transformation (11) the covariantly chiral superfield $\mathcal{R}$ transforms as

$$\mathcal{R} \to e^{-2\Phi}\left(\mathcal{R} - \frac{1}{4}\bar{\nabla}^2\right)e^{\bar{\Phi}} \ . \tag{12}$$

The super-Weyl chiral superfield parameter Φ can be traded for the chiral Lagrange multiplier $\mathcal{Z}$ by using a generic gauge condition

$$\mathcal{Z} = \mathcal{Z}(\Phi) \tag{13}$$

where $\mathcal{Z}(\Phi)$ is a holomorphic function of Φ. It results in the equivalent action

$$S_{\Phi} = \int d^4x d^4\theta\, E^{-1}e^{\Phi+\bar{\Phi}}\left[\mathcal{Z}(\Phi) + \text{H.c.}\right] - \int d^4x d^2\theta\, \mathcal{E}e^{3\Phi}V(\mathcal{Z}(\Phi)) + \text{H.c.} \tag{14}$$

Equation (14) has the standard form of the action of a chiral matter superfield coupled to supergravity,

$$S[\Phi,\bar{\Phi}] = \int d^4x d^4\theta\, E^{-1}\Omega(\Phi,\bar{\Phi}) + \left[\int d^4x d^2\theta\, \mathcal{E} P(\Phi) + \text{H.c.}\right] \ , \qquad (15)$$

in terms of a 'Kähler' potential $\Omega(\Phi,\bar{\Phi})$ and a chiral superpotential $P(\Phi)$. In our case (14) we find

$$\Omega(\Phi,\bar{\Phi}) = e^{\Phi+\bar{\Phi}}\left[\mathcal{Z}(\Phi) + \bar{\mathcal{Z}}(\bar{\Phi})\right] \ , \quad P(\Phi) = -e^{3\Phi}V(\mathcal{Z}(\Phi)) \ . \qquad (16)$$

The truly Kähler potential $K(\Phi,\bar{\Phi})$ is given by

$$K = -3\ln(-\frac{\Omega}{3}) \quad \text{or} \quad \Omega = -3e^{-K/3} \ , \qquad (17)$$

because of the invariance of the action (15) under the supersymmetric Kähler-Weyl transformations

$$K(\Phi,\bar{\Phi}) \to K(\Phi,\bar{\Phi}) + \Lambda(\Phi) + \bar{\Lambda}(\bar{\Phi}) \ , \quad \mathcal{E} \to e^{\Lambda(\Phi)}\mathcal{E} \ , \qquad (18)$$

$P(\Phi) \to -e^{-\Lambda(\Phi)}P(\Phi)$, with an arbitrary chiral superfield parameter $\Lambda(\Phi)$.

Scalar potential

(in components) is given by the standard formula (Cremmer *et al*, 1979)

$$\mathcal{V}(\phi,\bar{\phi}) = e^{\Omega}\left\{\left|\frac{\partial P}{\partial\Phi} + \frac{\partial\Omega}{\partial\Phi}P\right|^2 - 3\,|P|^2\right\}\Bigg| \tag{19}$$

where all superfields are restricted to their leading field components, $\Phi| = \phi(x)$. Equation (19) can be simplified by making use of the Kähler-Weyl invariance (18) that allows us to choose the gauge

$$P = 1 \tag{20}$$

It is equivalent to the well known fact that the scalar potential (19) is actually governed by the single (Kähler-Weyl-invariant) potential

$$G(\Phi,\bar{\Phi}) = \Omega + \ln|P|^2 \tag{21}$$

In our case (16) we have

$$G = e^{\Phi+\bar{\Phi}}\left[\mathcal{Z}(\Phi) + \bar{\mathcal{Z}}(\bar{\Phi})\right] + 3(\Phi+\bar{\Phi}) + \ln(V(\mathcal{Z}(\Phi)) + \ln(\bar{V}(\bar{\mathcal{Z}}(\bar{\Phi})) \tag{22}$$

Let's now specify our gauge (13) by choosing the condition

$$3\Phi + \ln(V(\mathcal{Z}(\Phi)) = 0 \quad \text{or} \quad V(\mathcal{Z}(\Phi)) = e^{-3\Phi} \tag{23}$$

that is equivalent to eq. (20). Then the potential (22) gets simplified to

$$G = \Omega = e^{\Phi+\bar{\Phi}} \left[\mathcal{Z}(\Phi) + \bar{\mathcal{Z}}(\bar{\Phi})\right] \tag{24}$$

Equations (8), (9) and (24) are the one-to-one relations between a holomorphic function $F(\mathcal{R})$ in the supergravity action (5) and a holomorphic function $\mathcal{Z}(\Phi)$ defining the scalar potential (19)

$$\mathcal{V} = e^{G}\left[\left(\frac{\partial^2 G}{\partial\Phi\partial\bar{\Phi}}\right)^{-1}\frac{\partial G}{\partial\Phi}\frac{\partial G}{\partial\bar{\Phi}} - 3\right]\bigg| \tag{25}$$

in the classically equivalent scalar-tensor supergravity.

Getting fields from superfields

Applying the superspace chiral density formula

$$\int d^4x d^2\theta\, \mathcal{E}\mathcal{L} = \int d^4x\, e \left\{\mathcal{L}_{\text{last}} + B\mathcal{L}_{\text{first}}\right\} \tag{26}$$

to our action (5) yields its bosonic part as

$$F'(\bar{X}) \left[\tfrac{1}{3}R_* + 4\bar{X}X\right] + 3XF(\bar{X}) + \text{H.c.} \tag{27}$$

where primes denote differentiation. We have used the notation

$$X = \tfrac{1}{3}B\ , \qquad R_* = R + \frac{i}{2}\varepsilon^{abcd}R_{abcd} \tag{28}$$

Varying eq. (27) with respect to the auxiliary fields X and $\bar{X}$ gives rise to merely algebraic equation on the auxiliary fields,

$$3\bar{F} + X(4\bar{F}' + 7F') + 4\bar{X}XF'' + \tfrac{1}{3}F''R_* = 0 \tag{29}$$

Example #1: Recovering pure Supergravity

- Let's consider the simple special case when

$$F'' = 0 \qquad \text{or, equivalently,} \qquad F(\mathcal{R}) = f_0 + f_1 \mathcal{R} \tag{30}$$

with some complex constants f_0 and f_1, where $\mathrm{Re} f_1 < 0$. Then eq. (29) is easily solved as

$$\bar{X} = \frac{-3(f_0 + f_1 R_*)}{4f_1 + 7\bar{f}_1} \tag{31}$$

Substituting the solution (31) back into the Lagrangian (27) yields

$$\frac{2}{3}(\mathrm{Re} f_1)R_* - \frac{9\,|f_0|^2}{14(\mathrm{Re} f_1)} \equiv -\frac{1}{2\kappa^2}R_* - \Lambda = -\frac{1}{2\kappa^2}R(\Gamma + T) - \Lambda \tag{32}$$

where we have reintroduced the standard gravitational constant $\kappa_0 = M^{-1}_{\mathrm{Planck}}$ in terms of the (reduced) Planck mass, the standard supergravity connection (i.e. Christoffel symbols Γ plus torsion T), and a cosmological constant Λ,

$$\kappa = \sqrt{\frac{3}{4\,|\mathrm{Re} f_1|}} \quad , \qquad \Lambda = \frac{-9\,|f_0|^2}{14\,|\mathrm{Re} f_1|} \tag{33}$$

Example #2: beyond pure supergravity

Let's consider the *Ansatz*

$$F(\mathcal{R}) = -\frac{1}{2} f_1 \mathcal{R} + \frac{1}{2} f_2 \mathcal{R}^2 \tag{34}$$

with some real constants f_1 and f_2, where the first term represents the standard (pure) $N = 1$ supergravity and the second term is a 'quantum correction'. We set $\psi_\mu = 0$, which also implies $R_* = R$ and the real auxiliary field X. We find

$$L_{\text{bos}} = 11 f_2 X^3 - 7 f_1 X^2 + \frac{2}{3} f_2 R X - \frac{1}{3} f_1 R \tag{35}$$

In the limit of $f_2 \to 0$ we thus have $X = 0$, as it should. Hence, we recover the Einstein-Hilbert Lagrangian

$$L_{\text{EH}} = -\frac{1}{3} f_1 R = -\frac{1}{2\kappa^2} R = -\frac{M_{\text{Pl}}^2}{2} R \tag{36}$$

provided that

$$f_1 = \frac{3}{2} M_{\text{Pl}}^2 \, . \qquad \text{Let's also use} \quad f_2 = \frac{M_{\text{Pl}}^2}{m} \, . \tag{37}$$

The auxiliary field equation (29) takes the form of a quadratic equation,

$$11X^2 - 7mX + \frac{2}{9}R = 0 \tag{38}$$

whose solution is given by

$$X_{\pm} = \frac{7m}{22}\left[1 \pm \sqrt{1 - \frac{8 \cdot 11R}{3^2 \cdot 7^2 m^2}}\right] = \left(\frac{2R_{\mathrm{max}}}{99}\right)^{1/2}\left[1 \pm \sqrt{1 - \frac{R}{R_{\mathrm{max}}}}\right] \tag{39}$$

where we have introduced the maximal scalar curvature

$$R_{\mathrm{max}} = \frac{99}{2}\left[\frac{7m}{22}\right]^2 \tag{40}$$

The existence of the built-in maximal (upper) scalar curvature (or the AdS bound) is a nice bonus of our construction. It is similar to the factor $\sqrt{1 - v^2/c^2}$ of Special Relativity. Yet another close analogy comes from the Born-Infeld non-linear extension of Maxwell electrodynamics, whose (dual) Hamiltonian is proportional to $\left(1 - \sqrt{1 - \vec{E}^2/E^2_{\mathrm{max}} - \vec{H}^2/H^2_{\mathrm{max}} + (\vec{E} \times \vec{H})^2/E^2_{\mathrm{max}}H^2_{\mathrm{max}}}\right)$ in terms of the electric and magnetic fields $\vec{E}$ and $\vec{H}$, respectively, with their maximal values. For instance, in string theory, one has $E_{\mathrm{max}} = H_{\mathrm{max}} = (2\pi\alpha')^{-1}$.

Special $f(R)$-gravity from $F(R)$-Supergravity

Equation (38) can be used to reduce the Lagrangian (35) to a linear function of X by double iteration. Then a substitution of the solution (39) into the Lagrangian gives us a bosonic $f(R)$ gravity Lagrangian (6) in the form

$$f_{\pm}(R) = \frac{-5 \cdot 17 M_{\mathrm{Pl}}^2}{2 \cdot 3^2 \cdot 11} R + \frac{2 \cdot 7}{3^2 \cdot 11} M_{\mathrm{Pl}}^2 \left(R - R_{\mathrm{max}}\right) \left[1 \pm \sqrt{1 - R/R_{\mathrm{max}}} \right] \tag{41}$$

By construction, in the limit $m \to +\infty$ (or $R_{\mathrm{max}} \to +\infty$) both functions $f_{\pm}$ reproduce General Relativity. In another limit $R \to 0$, we find a cosmological constant,

$$f_{-}(0) \equiv \Lambda_{-} = 0 \, , \qquad f_{+}(0) \equiv \Lambda_{+} = -\frac{7^3}{2^2 \cdot 11^2} M_{\mathrm{Pl}}^2 m^2 = -\frac{14}{99} M_{\mathrm{Pl}}^2 R_{\mathrm{max}} \tag{42}$$

Example #2: Scalar potential and inflation

In the case of the supergravity-generated function $f_{-}(R)$, the inflaton scalar potential reads

$$V(y) = \frac{3^3}{2^6} M_{\rm Pl}^2 m^2 \left(11e^y + 3\right)\left(e^{-y} - 1\right)^2$$

The last factor $(e^{-y} - 1)^2$ of this potential is the same as that of the Starobinsky model. However, the extra factor $(11e^y + 3)$ does not allow for a slow-roll inflation because of

$$\varepsilon(y) = \frac{1}{3}\left[\frac{e^y\left(11 + 11e^{-y} + 6e^{-2y}\right)}{(11e^y + 3)(e^{-y} - 1)}\right]^2 \geq \frac{1}{3} \tag{43}$$

and

$$\eta(y) = \frac{2}{3}\frac{\left(11e^y + 5e^{-y} + 12e^{-2y}\right)}{(11e^y + 3)(e^{-y} - 1)^2} \geq \frac{2}{3} \tag{44}$$

#3: Viable embedding of Starobinsky model into Supergravity

$$\text{Let}'\text{s add a cubic term}: \quad F(\mathcal{R}) = -\frac{1}{2}f_1\mathcal{R} + \frac{1}{2}f_2\mathcal{R}^2 - \frac{1}{6}f_3\mathcal{R}^3$$

with some real positive coefficients f_1, f_2 and f_3. The auxiliary field equation reads

$$X^3 - \left(\frac{33f_2}{20f_3}\right)X^2 + \left(\frac{7f_1}{10f_3} + \frac{1}{30}R\right)X - \frac{f_2}{30f_3}R = 0$$

There are 3 different regimes. In part, when $f_2^2 \ll f_1 f_3$, we find a simple solution

$$X^2 = -\frac{1}{30}(R + R_0)$$

when $R < -R_0$ with $R_0 = 21f_1/f_3$, which gives rise to the bosonic Lagrangian

$$L_{\text{bos}} = -\frac{f_1}{3}R + \frac{f_3}{180}(R + R_0)^2$$

Thus, for large curvatures $|R| \gg R_0$ one gets the Starobinsky model! To get this embedding viable for chaotic inflation, one also needs large f_3 (to get $|R| \ll M_{\text{Pl}}$) and large f_2^2/f_1 (to get $m_{\text{inf}} \ll M_{\text{Pl}}$).

Conclusion

- A manifestly $4D$, $N=1$ supersymmetric extension of $f(R)$ gravity exist, it is chiral and is parametrized by a holomorphic function. An $F(R)$ supergravity is classically equivalent to the standard theory of a chiral scalar superfield (with certain Kähler potential and a superpotential) miminally coupled to the $N=1$ Poincaré supergravity in four spacetime dimensions.

- The inflaton scalar potential is derivable via the (non-perturbative) Legendre–Kähler-Weyl transform in superspace, and is governed by a single holomorphic function. The Starobinsky model of chaotic inflation can be embedded into $F(\mathcal{R})$ supergravity. The $F(R)$ supergravity predicts the existence of the maximal scalar curvature, or AdS (upper) bound.

- We conjectured an identification of the dynamical chiral superfield in $F(R)$ supergravity with the dilaton-axion chiral superfield in 4D Superstring Theory. The $R^2A(R)$ terms may appear in the gravitational effective action after superstring compactification, the problem is to get a large coefficient.