Submanifolds and Holonomy

Carlos Olmos

FaMAF - UNC

Tokyo, November, 2009

Introduction.

Riemannian holonomy.

Submanifold geometry and holonomy.

Holonomy systems and skew-torsion holonomy systems.

Applications to naturally reductive spaces.

1. Introduction.
2. Riemannian holonomy.
3. Submanifold geometry and holonomy.
4. Holonomy systems and skew-torsion holonomy systems.
5. Applications to naturally reductive spaces.

Introduction.

Symmetric spaces play a central rôle in Riemannian geometry.

On one hand, this family of homogeneous spaces is so wide that studying the geometry of such spaces one can infer general results in Riemannian geometry, e.g. de Rham decompositon theorem or properties of spaces with sectional curvatures of the same sign.
On the other hand, this family is so particular, that the geometry of symmetric is very rich and non-generic.

Some of the more beautiful and important results in Riemannian geometry, are theorems that under mild non-generic assumptions imply symmetry.This is the case of the Berger holonomy theorem and the rank rigidity theorem of Ballmann/Burns-Spatzier.

Introduction.

Symmetric spaces play a central rôle in Riemannian geometry.
On one hand, this family of homogeneous spaces is so wide that studying the geometry of such spaces one can infer general results in Riemannian geometry, e.g. de Rham decompositon theorem or properties of spaces with sectional curvatures of the same sign.
On the other hand, this family is so particular, that the geometry of symmetric is very rich and non-generic.

Some of the more beautiful and important results in Riemannian geometry, are theorems that under mild non-generic assumptions imply symmetry.This is the case of the Berger holonomy theorem and the rank rigidity theorem of Ballmann/Burns-Spatzier.

Introduction.

Riemannian holonomy.

Submanifold geometry and holonomy.

Holonomy systems and skew-torsion holonomy systems.

Applications to naturally reductive spaces.

Introduction.

Symmetric spaces play a central rôle in Riemannian geometry.
On one hand, this family of homogeneous spaces is so wide that studying the geometry of such spaces one can infer general results in Riemannian geometry, e.g. de Rham decompositon theorem or properties of spaces with sectional curvatures of the same sign.
On the other hand, this family is so particular, that the geometry of symmetric is very rich and non-generic.

Some of the more beautiful and important results in Riemannian geometry, are theorems that under mild non-generic assumptions imply symmetry.This is the case of the Berger holonomy theorem and the rank rigidity theorem of Ballmann/Burns-Spatzier.

Introduction.

Symmetric spaces play a central rôle in Riemannian geometry.
On one hand, this family of homogeneous spaces is so wide that studying the geometry of such spaces one can infer general results in Riemannian geometry, e.g. de Rham decompositon theorem or properties of spaces with sectional curvatures of the same sign.
On the other hand, this family is so particular, that the geometry of symmetric is very rich and non-generic.

Some of the more beautiful and important results in Riemannian geometry, are theorems that under mild non-generic assumptions imply symmetry.This is the case of the Berger holonomy theorem and the rank rigidity theorem of Ballmann/Burns-Spatzier.

Introduction.

Riemannian holonomy.

Submanifold geometry and holonomy.

Holonomy systems and skew-torsion holonomy systems.

Applications to naturally reductive spaces.

Introduction.

Symmetric spaces play a central rôle in Riemannian geometry.
On one hand, this family of homogeneous spaces is so wide that studying the geometry of such spaces one can infer general results in Riemannian geometry, e.g. de Rham decompositon theorem or properties of spaces with sectional curvatures of the same sign.
On the other hand, this family is so particular, that the geometry of symmetric is very rich and non-generic.

Some of the more beautiful and important results in Riemannian geometry, are theorems that under mild non-generic assumptions imply symmetry.This is the case of the Berger holonomy theorem and the rank rigidity theorem of Ballmann/Burns-Spatzier.

Introduction.

Symmetric spaces play a central rôle in Riemannian geometry.
On one hand, this family of homogeneous spaces is so wide that studying the geometry of such spaces one can infer general results in Riemannian geometry, e.g. de Rham decompositon theorem or properties of spaces with sectional curvatures of the same sign.
On the other hand, this family is so particular, that the geometry of symmetric is very rich and non-generic.

Some of the more beautiful and important results in Riemannian geometry, are theorems that under mild non-generic assumptions imply symmetry.This is the case of the Berger holonomy theorem and the rank rigidity theorem of Ballmann/Burns-Spatzier.

Introduction.

Symmetric spaces play a central rôle in Riemannian geometry.
On one hand, this family of homogeneous spaces is so wide that studying the geometry of such spaces one can infer general results in Riemannian geometry, e.g. de Rham decompositon theorem or properties of spaces with sectional curvatures of the same sign.
On the other hand, this family is so particular, that the geometry of symmetric is very rich and non-generic.

Some of the more beautiful and important results in Riemannian geometry, are theorems that under mild non-generic assumptions imply symmetry. This is the case of the Berger holonomy theorem and the rank rigidity theorem of Ballmann/Burns-Spatzier.

Introduction.

Symmetric spaces play a central rôle in Riemannian geometry.
On one hand, this family of homogeneous spaces is so wide that studying the geometry of such spaces one can infer general results in Riemannian geometry, e.g. de Rham decompositon theorem or properties of spaces with sectional curvatures of the same sign.
On the other hand, this family is so particular, that the geometry of symmetric is very rich and non-generic.

Some of the more beautiful and important results in Riemannian geometry, are theorems that under mild non-generic assumptions imply symmetry.This is the case of the Berger holonomy theorem and the rank rigidity theorem of Ballmann/Burns-Spatzier.

A similar rôle, that symmetric spaces play in Riemannian geometry, is played, in Euclidean submanifold geometry, by the orbits of s-representations (i.e., isotropy representations of semisimple symmetric spaces).

Some remarkable results in submanifold geometry are theorems that imply that a given non-generic submanifold is an orbit of an s-representation.

This is the case of the well-known theorem of Thorbergsson about the homogeneity of isoparametric submanifolds of high codimension. This is also the case of the so-called Rank Rigidity theorem for submanifolds (O., Di Scala-O.)

The above mentioned results on Riemannian geometry look similar to those on submanifold geometry (being the main ingredients Riemannian or normal holonomy).

A similar rôle, that symmetric spaces play in Riemannian geometry, is played, in Euclidean submanifold geometry, by the orbits of s-representations (i.e., isotropy representations of semisimple symmetric spaces).

Some remarkable results in submanifold geometry are theorems that imply that a given non-generic submanifold is an orbit of an s-representation.

This is the case of the well-known theorem of Thorbergsson about the homogeneity of isoparametric submanifolds of high codimension. This is also the case of the so-called Rank Rigidity theorem for submanifolds (O., Di Scala-O.)

The above mentioned results on Riemannian geometry look similar to those on submanifold geometry (being the main ingredients Riemannian or normal holonomy).

A similar rôle, that symmetric spaces play in Riemannian geometry, is played, in Euclidean submanifold geometry, by the orbits of s-representations (i.e., isotropy representations of semisimple symmetric spaces).

Some remarkable results in submanifold geometry are theorems that imply that a given non-generic submanifold is an orbit of an s-representation.

This is the case of the well-known theorem of Thorbergsson about the homogeneity of isoparametric submanifolds of high codimension. This is also the case of the so-called Rank Rigidity theorem for submanifolds (O., Di Scala-O.)

The above mentioned results on Riemannian geometry look similar to those on submanifold geometry (being the main ingredients Riemannian or normal holonomy).

A similar rôle, that symmetric spaces play in Riemannian geometry, is played, in Euclidean submanifold geometry, by the orbits of s-representations (i.e., isotropy representations of semisimple symmetric spaces).

Some remarkable results in submanifold geometry are theorems that imply that a given non-generic submanifold is an orbit of an s-representation.

This is the case of the well-known theorem of Thorbergsson about the homogeneity of isoparametric submanifolds of high codimension. This is also the case of the so-called Rank Rigidity theorem for submanifolds (O., Di Scala-O.)

The above mentioned results on Riemannian geometry look similar to those on submanifold geometry (being the main ingredients Riemannian or normal holonomy).

A similar rôle, that symmetric spaces play in Riemannian geometry, is played, in Euclidean submanifold geometry, by the orbits of s-representations (i.e., isotropy representations of semisimple symmetric spaces).

Some remarkable results in submanifold geometry are theorems that imply that a given non-generic submanifold is an orbit of an s-representation.

This is the case of the well-known theorem of Thorbergsson about the homogeneity of isoparametric submanifolds of high codimension. This is also the case of the so-called Rank Rigidity theorem for submanifolds (O., Di Scala-O.)

The above mentioned results on Riemannian geometry look similar to those on submanifold geometry (being the main ingredients Riemannian or normal holonomy).

A similar rôle, that symmetric spaces play in Riemannian geometry, is played, in Euclidean submanifold geometry, by the orbits of s-representations (i.e., isotropy representations of semisimple symmetric spaces).

Some remarkable results in submanifold geometry are theorems that imply that a given non-generic submanifold is an orbit of an s-representation.

This is the case of the well-known theorem of Thorbergsson about the homogeneity of isoparametric submanifolds of high codimension. This is also the case of the so-called Rank Rigidity theorem for submanifolds (O., Di Scala-O.)

The above mentioned results on Riemannian geometry look similar to those on submanifold geometry (being the main ingredients Riemannian or normal holonomy).

A similar rôle, that symmetric spaces play in Riemannian geometry, is played, in Euclidean submanifold geometry, by the orbits of s-representations (i.e., isotropy representations of semisimple symmetric spaces).

Some remarkable results in submanifold geometry are theorems that imply that a given non-generic submanifold is an orbit of an s-representation.

This is the case of the well-known theorem of Thorbergsson about the homogeneity of isoparametric submanifolds of high codimension. This is also the case of the so-called Rank Rigidity theorem for submanifolds (O., Di Scala-O.)

The above mentioned results on Riemannian geometry look similar to those on submanifold geometry (being the main ingredients Riemannian or normal holonomy).

A similar rôle, that symmetric spaces play in Riemannian geometry, is played, in Euclidean submanifold geometry, by the orbits of s-representations (i.e., isotropy representations of semisimple symmetric spaces).

Some remarkable results in submanifold geometry are theorems that imply that a given non-generic submanifold is an orbit of an s-representation.

This is the case of the well-known theorem of Thorbergsson about the homogeneity of isoparametric submanifolds of high codimension. This is also the case of the so-called Rank Rigidity theorem for submanifolds (O., Di Scala-O.)

The above mentioned results on Riemannian geometry look similar to those on submanifold geometry (being the main ingredients Riemannian or normal holonomy).

A similar rôle, that symmetric spaces play in Riemannian geometry, is played, in Euclidean submanifold geometry, by the orbits of s-representations (i.e., isotropy representations of semisimple symmetric spaces).

Some remarkable results in submanifold geometry are theorems that imply that a given non-generic submanifold is an orbit of an s-representation.

This is the case of the well-known theorem of Thorbergsson about the homogeneity of isoparametric submanifolds of high codimension. This is also the case of the so-called Rank Rigidity theorem for submanifolds (O., Di Scala-O.)

The above mentioned results on Riemannian geometry look similar to those on submanifold geometry (being the main ingredients Riemannian or normal holonomy).

This would suggests that there is a subtle relation between this two contexts.

This is explained, in part, by the fact that using submanifold geometry, with normal holonomy ingredients, one can prove, geometrically, the Berger holonomy theorem (O., Ann. of Math. 2005), or the Simons (algebraic) holonomy theorem (O., L' Enseign.Math., 2005).

We will begin surveying on the above mentioned topics.

In the last part, we will refer to a recent new application of Euclidean submanifold geometry to Riemannian homogeneous geometry: the so-called Skew-torsion holonomy theorem (O.-Reggiani).

This would suggests that there is a subtle relation between this two contexts.

This is explained, in part, by the fact that using submanifold geometry, with normal holonomy ingredients, one can prove, geometrically, the Berger holonomy theorem (O., Ann. of Math. 2005), or the Simons (algebraic) holonomy theorem (O., L' Enseign.Math., 2005).

We will begin surveying on the above mentioned topics.

In the last part, we will refer to a recent new application of Euclidean submanifold geometry to Riemannian homogeneous geometry: the so-called Skew-torsion holonomy theorem (O.-Reggiani).

This would suggests that there is a subtle relation between this two contexts.

This is explained, in part, by the fact that using submanifold geometry, with normal holonomy ingredients, one can prove, geometrically, the Berger holonomy theorem (O., Ann. of Math. 2005), or the Simons (algebraic) holonomy theorem (O., L' Enseign.Math., 2005).

We will begin surveying on the above mentioned topics.

In the last part, we will refer to a recent new application of Euclidean submanifold geometry to Riemannian homogeneous geometry: the so-called Skew-torsion holonomy theorem (O.-Reggiani).

This would suggests that there is a subtle relation between this two contexts.

This is explained, in part, by the fact that using submanifold geometry, with normal holonomy ingredients, one can prove, geometrically, the Berger holonomy theorem (O., Ann. of Math. 2005), or the Simons (algebraic) holonomy theorem (O., L' Enseign.Math., 2005).

We will begin surveying on the above mentioned topics.

In the last part, we will refer to a recent new application of Euclidean submanifold geometry to Riemannian homogeneous geometry: the so-called Skew-torsion holonomy theorem (O.-Reggiani).

This would suggests that there is a subtle relation between this two contexts.

This is explained, in part, by the fact that using submanifold geometry, with normal holonomy ingredients, one can prove, geometrically, the Berger holonomy theorem (O., Ann. of Math. 2005), or the Simons (algebraic) holonomy theorem (O., L' Enseign.Math., 2005).

We will begin surveying on the above mentioned topics.

In the last part, we will refer to a recent new application of Euclidean submanifold geometry to Riemannian homogeneous geometry: the so-called Skew-torsion holonomy theorem (O.-Reggiani).

This would suggests that there is a subtle relation between this two contexts.

This is explained, in part, by the fact that using submanifold geometry, with normal holonomy ingredients, one can prove, geometrically, the Berger holonomy theorem (O., Ann. of Math. 2005), or the Simons (algebraic) holonomy theorem (O., L' Enseign.Math., 2005).

We will begin surveying on the above mentioned topics.

In the last part, we will refer to a recent new application of Euclidean submanifold geometry to Riemannian homogeneous geometry: the so-called Skew-torsion holonomy theorem (O.-Reggiani).

Riemannian holonomy.

If M is a Riemannian manifold and $E = TM$, then the de Rham decomposition theorem implies that the (restricted) holonomy group acts irreducibly, unless M is a Riemannian product (locally, or globally if M is simply connected and complete).

Moreover, one has the well-known result of Marcel Berger (1955).

Berger Holonomy Theorem. *Assume that the (restricted) holonomy group of an irreducible Riemannian manifold is not transitive on the sphere. Then M is locally symmetric.*

The above theorem follows from the classification of Berger of the possible holonomies of locally non-symmetric spaces. Namely,

Riemannian holonomy.

If M is a Riemannian manifold and $E = TM$, then the de Rham decomposition theorem implies that the (restricted) holonomy group acts irreducibly, unless M is a Riemannian product (locally, or globally if M is simply connected and complete).

Moreover, one has the well-known result of Marcel Berger (1955).

Berger Holonomy Theorem. *Assume that the (restricted) holonomy group of an irreducible Riemannian manifold is not transitive on the sphere. Then M is locally symmetric.*

The above theorem follows from the classification of Berger of the possible holonomies of locally non-symmetric spaces. Namely,

Riemannian holonomy.

If M is a Riemannian manifold and $E = TM$, then the de Rham decomposition theorem implies that the (restricted) holonomy group acts irreducibly, unless M is a Riemannian product (locally, or globally if M is simply connected and complete).

Moreover, one has the well-known result of Marcel Berger (1955).

Berger Holonomy Theorem. *Assume that the (restricted) holonomy group of an irreducible Riemannian manifold is not transitive on the sphere. Then M is locally symmetric.*

The above theorem follows from the classification of Berger of the possible holonomies of locally non-symmetric spaces. Namely,

Introduction.

Riemannian holonomy.

Submanifold geometry and holonomy.

Holonomy systems and skew-torsion holonomy systems.

Applications to naturally reductive spaces.

Riemannian holonomy.

If M is a Riemannian manifold and $E = TM$, then the de Rham decomposition theorem implies that the (restricted) holonomy group acts irreducibly, unless M is a Riemannian product (locally, or globally if M is simply connected and complete).

Moreover, one has the well-known result of Marcel Berger (1955).

Berger Holonomy Theorem. *Assume that the (restricted) holonomy group of an irreducible Riemannian manifold is not transitive on the sphere. Then M is locally symmetric.*

The above theorem follows from the classification of Berger of the possible holonomies of locally non-symmetric spaces. Namely,

- *$SO(n)$, generic Riemannian manifold* of dimension n.
- *$U(n)$, generic Khäler manifold* of real dimension $2n$.
- *$Sp(n) \times Sp(1)$, generic quaternionic Khäler manifold*, $\dim = 4n$ (always Einstein).
- *$Spin(9)$*, always symmetric, *the Cayley plane or its dual*, $\dim = 16$ (Dmitri Alekseevsky).
- *$SU(n)$, Calabi-Yau manifolds*, $\dim = 2n$ (Khäler)
- *$Sp(n)$*, generic hyperkhäler manifold, $\dim = 4n$ (Khäler and Ricci flat) .
- G_2, the so-called *G_2-manifolds*, $\dim = 7$ (Ricci flat).
- *$Spin(7)$*, the so-called *$Spin(7)$-manifolds*, $\dim = 8$ (Ricci flat).

The last four groups in the list are the so-called exceptional holonomies, since there are no symmetric spaces having them as holonomy.

- $SO(n)$, *generic Riemannian manifold* of dimension n.
- $U(n)$, *generic Khäler manifold* of real dimension $2n$.
- $Sp(n) \times Sp(1)$, *generic quaternionic Khäler manifold*, $\dim = 4n$ (always Einstein).
- $Spin(9)$, always symmetric, *the Cayley plane or its dual*, $\dim = 16$ (Dmitri Alekseevsky).
- $SU(n)$, *Calabi-Yau manifolds*, $\dim = 2n$ (Khäler)
- $Sp(n)$, generic hyperkhäler manifold, $\dim = 4n$ (Khäler and Ricci flat) .
- G_2, the so-called G_2-*manifolds*, $\dim = 7$ (Ricci flat).
- $Spin(7)$, the so-called $Spin(7)$-*manifolds*, $\dim = 8$ (Ricci flat).

The last four groups in the list are the so-called exceptional holonomies, since there are no symmetric spaces having them as holonomy.

- $SO(n)$, *generic Riemannian manifold* of dimension n.
- $U(n)$, *generic Khäler manifold* of real dimension $2n$.
- $Sp(n) \times Sp(1)$, *generic quaternionic Khäler manifold*, $\dim = 4n$ (always Einstein).
- $Spin(9)$, always symmetric, *the Cayley plane or its dual*, $\dim = 16$ (Dmitri Alekseevsky).
- $SU(n)$, *Calabi-Yau manifolds*, $\dim = 2n$ (Khäler)
- $Sp(n)$, generic hyperkhäler manifold, $\dim = 4n$ (Khäler and Ricci flat) .
- G_2, the so-called *G_2-manifolds*, $\dim = 7$ (Ricci flat).
- $Spin(7)$, the so-called *$Spin(7)$-manifolds*, $\dim = 8$ (Ricci flat).

The last four groups in the list are the so-called exceptional holonomies, since there are no symmetric spaces having them as holonomy.

- $SO(n)$, *generic Riemannian manifold* of dimension n.
- $U(n)$, *generic Khäler manifold* of real dimension $2n$.
- $Sp(n) \times Sp(1)$, *generic quaternionic Khäler manifold*, dim $= 4n$ (always Einstein).
- $Spin(9)$, always symmetric, *the Cayley plane or its dual*, dim $= 16$ (Dmitri Alekseevsky).
- $SU(n)$, *Calabi-Yau manifolds*, dim $= 2n$ (Khäler)
- $Sp(n)$, generic hyperkhäler manifold, dim $= 4n$ (Khäler and Ricci flat) .
- G_2, the so-called G_2*-manifolds*, dim $= 7$ (Ricci flat).
- $Spin(7)$, the so-called $Spin(7)$*-manifolds*, dim $= 8$ (Ricci flat).

The last four groups in the list are the so-called exceptional holonomies, since there are no symmetric spaces having them as holonomy.

- $SO(n)$, *generic Riemannian manifold* of dimension n.
- $U(n)$, *generic Khäler manifold* of real dimension $2n$.
- $Sp(n) \times Sp(1)$, *generic quaternionic Khäler manifold*, $\dim = 4n$ (always Einstein).
- $Spin(9)$, always symmetric, *the Cayley plane or its dual*, $\dim = 16$ (Dmitri Alekseevsky).
- $SU(n)$, *Calabi-Yau manifolds*, $\dim = 2n$ (Khäler)
- $Sp(n)$, generic hyperkhäler manifold, $\dim = 4n$ (Khäler and Ricci flat) .
- G_2, the so-called G_2-*manifolds*, $\dim = 7$ (Ricci flat).
- $Spin(7)$, the so-called $Spin(7)$-*manifolds*, $\dim = 8$ (Ricci flat).

The last four groups in the list are the so-called exceptional holonomies, since there are no symmetric spaces having them as holonomy.

- $SO(n)$, *generic Riemannian manifold* of dimension n.
- $U(n)$, *generic Khäler manifold* of real dimension $2n$.
- $Sp(n) \times Sp(1)$, *generic quaternionic Khäler manifold*, $\dim = 4n$ (always Einstein).
- $Spin(9)$, always symmetric, *the Cayley plane or its dual*, $\dim = 16$ (Dmitri Alekseevsky).
- $SU(n)$, *Calabi-Yau manifolds*, $\dim = 2n$ (Khäler)
- $Sp(n)$, generic hyperkhäler manifold, $\dim = 4n$ (Khäler and Ricci flat) .
- G_2, the so-called G_2-*manifolds*, $\dim = 7$ (Ricci flat).
- $Spin(7)$, the so-called $Spin(7)$-*manifolds*, $\dim = 8$ (Ricci flat).

The last four groups in the list are the so-called exceptional holonomies, since there are no symmetric spaces having them as holonomy.

- $SO(n)$, *generic Riemannian manifold* of dimension n.
- $U(n)$, *generic Khäler manifold* of real dimension $2n$.
- $Sp(n) \times Sp(1)$, *generic quaternionic Khäler manifold*, $\dim = 4n$ (always Einstein).
- $Spin(9)$, always symmetric, *the Cayley plane or its dual*, $\dim = 16$ (Dmitri Alekseevsky).
- $SU(n)$, *Calabi-Yau manifolds*, $\dim = 2n$ (Khäler)
- $Sp(n)$, generic hyperkhäler manifold, $\dim = 4n$ (Khäler and Ricci flat) .
- G_2, the so-called G_2-*manifolds*, $\dim = 7$ (Ricci flat).
- $Spin(7)$, the so-called $Spin(7)$-*manifolds*, $\dim = 8$ (Ricci flat).

The last four groups in the list are the so-called exceptional holonomies, since there are no symmetric spaces having them as holonomy.

- $SO(n)$, *generic Riemannian manifold* of dimension n.
- $U(n)$, *generic Khäler manifold* of real dimension $2n$.
- $Sp(n) \times Sp(1)$, *generic quaternionic Khäler manifold*, $\dim = 4n$ (always Einstein).
- $Spin(9)$, always symmetric, *the Cayley plane or its dual*, $\dim = 16$ (Dmitri Alekseevsky).
- $SU(n)$, *Calabi-Yau manifolds*, $\dim = 2n$ (Khäler)
- $Sp(n)$, generic hyperkhäler manifold, $\dim = 4n$ (Khäler and Ricci flat) .
- G_2, the so-called G_2-*manifolds*, $\dim = 7$ (Ricci flat).
- $Spin(7)$, the so-called $Spin(7)$-*manifolds*, $\dim = 8$ (Ricci flat).

The last four groups in the list are the so-called exceptional holonomies, since there are no symmetric spaces having them as holonomy.

- $SO(n)$, *generic Riemannian manifold* of dimension n.
- $U(n)$, *generic Khäler manifold* of real dimension $2n$.
- $Sp(n) \times Sp(1)$, *generic quaternionic Khäler manifold*, $\dim = 4n$ (always Einstein).
- $Spin(9)$, always symmetric, *the Cayley plane or its dual*, $\dim = 16$ (Dmitri Alekseevsky).
- $SU(n)$, *Calabi-Yau manifolds*, $\dim = 2n$ (Khäler)
- $Sp(n)$, generic hyperkhäler manifold, $\dim = 4n$ (Khäler and Ricci flat) .
- G_2, the so-called *G_2-manifolds*, $\dim = 7$ (Ricci flat).
- $Spin(7)$, the so-called *$Spin(7)$-manifolds*, $\dim = 8$ (Ricci flat).

The last four groups in the list are the so-called exceptional holonomies, since there are no symmetric spaces having them as holonomy.

- $SO(n)$, *generic Riemannian manifold* of dimension n.
- $U(n)$, *generic Khäler manifold* of real dimension $2n$.
- $Sp(n) \times Sp(1)$, *generic quaternionic Khäler manifold*, $\dim = 4n$ (always Einstein).
- $Spin(9)$, always symmetric, *the Cayley plane or its dual*, $\dim = 16$ (Dmitri Alekseevsky).
- $SU(n)$, *Calabi-Yau manifolds*, $\dim = 2n$ (Khäler)
- $Sp(n)$, generic hyperkhäler manifold, $\dim = 4n$ (Khäler and Ricci flat) .
- G_2, the so-called G_2-*manifolds*, $\dim = 7$ (Ricci flat).
- $Spin(7)$, the so-called $Spin(7)$-*manifolds*, $\dim = 8$ (Ricci flat).

The last four groups in the list are the so-called exceptional holonomies, since there are no symmetric spaces having them as holonomy.

The only remaining group that is transitive on a sphere, but not a holonomy, is $Sp(n) \times S^1$ (which was excluded by Berger)

The idea of classifying geometries in terms of their parallel tensors or spinors (which can be read off from the holonomy) goes back to Lichnerowicz.

Marcel Berger succeed in giving such a classification, which turned out to be very important since associated to any holonomy there is a well-known and famous geometry!.

Introduction.

Riemannian holonomy.

Submanifold geometry and holonomy.

Holonomy systems and skew-torsion holonomy systems.

Applications to naturally reductive spaces.

Introduction.

Riemannian holonomy.

Submanifold geometry and holonomy.

Holonomy systems and skew-torsion holonomy systems.

Applications to naturally reductive spaces.

The only remaining group that is transitive on a sphere, but not a holonomy, is $Sp(n) \times S^1$ (which was excluded by Berger)

The idea of classifying geometries in terms of their parallel tensors or spinors (which can be read off from the holonomy) goes back to Lichnerowicz.

Marcel Berger succeed in giving such a classification, which turned out to be very important since associated to any holonomy there is a well-known and famous geometry!.

The only remaining group that is transitive on a sphere, but not a holonomy, is $Sp(n) \times S^1$ (which was excluded by Berger)

The idea of classifying geometries in terms of their parallel tensors or spinors (which can be read off from the holonomy) goes back to Lichnerowicz.

Marcel Berger succeed in giving such a classification, which turned out to be very important since associated to any holonomy there is a well-known and famous geometry!.

The only remaining group that is transitive on a sphere, but not a holonomy, is $Sp(n) \times S^1$ (which was excluded by Berger)

The idea of classifying geometries in terms of their parallel tensors or spinors (which can be read off from the holonomy) goes back to Lichnerowicz.

Marcel Berger succeed in giving such a classification, which turned out to be very important since associated to any holonomy there is a well-known and famous geometry!.

In 1962 James Simons succeed in giving a classification free proof of Berger theorem by defining the so-called holonomy systems. . This is triple

$$[\mathbb{V}, R, G]$$

where $\mathbb{V}$ is a Euclidean vector space, G is a Lie subgroup of $SO(\mathbb{V})$ and R is an algebraic Riemannian curvature tensor on $\mathbb{V}$ with values $R_{x,y} \in \mathcal{G}$.
Such a triple is said

irreducible, if G acts irreducibly on $\mathbb{V}$.
transitive, if G acts transitively on the unit sphere of $\mathbb{V}$.
symmetric, if $g(R) = R$, for all $g \in G$.

In 1962 James Simons succeed in giving a classification free proof of Berger theorem by defining the so-called holonomy systems. . This is triple

$$[\mathbb{V}, R, G]$$

where $\mathbb{V}$ is a Euclidean vector space, G is a Lie subgroup of $SO(\mathbb{V})$ and R is an algebraic Riemannian curvature tensor on $\mathbb{V}$ with values $R_{x,y} \in \mathcal{G}$.
Such a triple is said

irreducible, if G acts irreducibly on $\mathbb{V}$.
transitive, if G acts transitively on the unit sphere of $\mathbb{V}$.
symmetric, if $g(R) = R$, for all $g \in G$.

In 1962 James Simons succeed in giving a classification free proof of Berger theorem by defining the so-called holonomy systems. . This is triple

$$[\mathbb{V}, R, G]$$

where $\mathbb{V}$ is a Euclidean vector space, G is a Lie subgroup of $SO(\mathbb{V})$ and R is an algebraic Riemannian curvature tensor on $\mathbb{V}$ with values $R_{x,y} \in \mathcal{G}$.
Such a triple is said

irreducible, if G acts irreducibly on $\mathbb{V}$.
transitive, if G acts transitively on the unit sphere of $\mathbb{V}$.
symmetric, if $g(R) = R$, for all $g \in G$.

In 1962 James Simons succeed in giving a classification free proof of Berger theorem by defining the so-called holonomy systems. . This is triple

$$[\mathbb{V}, R, G]$$

where $\mathbb{V}$ is a Euclidean vector space, G is a Lie subgroup of $SO(\mathbb{V})$ and R is an algebraic Riemannian curvature tensor on $\mathbb{V}$ with values $R_{x,y} \in \mathcal{G}$.
Such a triple is said

irreducible, if G acts irreducibly on $\mathbb{V}$.
transitive, if G acts transitively on the unit sphere of $\mathbb{V}$.
symmetric, if $g(R) = R$, for all $g \in G$.

In 1962 James Simons succeed in giving a classification free proof of Berger theorem by defining the so-called holonomy systems. . This is triple

$$[\mathbb{V}, R, G]$$

where $\mathbb{V}$ is a Euclidean vector space, G is a Lie subgroup of $SO(\mathbb{V})$ and R is an algebraic Riemannian curvature tensor on $\mathbb{V}$ with values $R_{x,y} \in \mathcal{G}$.

Such a triple is said

irreducible, if G acts irreducibly on $\mathbb{V}$.

transitive, if G acts transitively on the unit sphere of $\mathbb{V}$.

symmetric, if $g(R) = R$, for all $g \in G$.

In 1962 James Simons succeed in giving a classification free proof of Berger theorem by defining the so-called holonomy systems. . This is triple

$$[\mathbb{V}, R, G]$$

where $\mathbb{V}$ is a Euclidean vector space, G is a Lie subgroup of $SO(\mathbb{V})$ and R is an algebraic Riemannian curvature tensor on $\mathbb{V}$ with values $R_{x,y} \in \mathcal{G}$.
Such a triple is said

irreducible, if G acts irreducibly on $\mathbb{V}$.
transitive, if G acts transitively on the unit sphere of $\mathbb{V}$.
symmetric, if $g(R) = R$, for all $g \in G$.

In 1962 James Simons succeed in giving a classification free proof of Berger theorem by defining the so-called holonomy systems. . This is triple

$$[\mathbb{V}, R, G]$$

where $\mathbb{V}$ is a Euclidean vector space, G is a Lie subgroup of $SO(\mathbb{V})$ and R is an algebraic Riemannian curvature tensor on $\mathbb{V}$ with values $R_{x,y} \in \mathcal{G}$.
Such a triple is said

irreducible, if G acts irreducibly on $\mathbb{V}$.

transitive, if G acts transitively on the unit sphere of $\mathbb{V}$.

symmetric, if $g(R) = R$, for all $g \in G$.

In 1962 James Simons succeed in giving a classification free proof of Berger theorem by defining the so-called holonomy systems. . This is triple

$$[\mathbb{V}, R, G]$$

where $\mathbb{V}$ is a Euclidean vector space, G is a Lie subgroup of $SO(\mathbb{V})$ and R is an algebraic Riemannian curvature tensor on $\mathbb{V}$ with values $R_{x,y} \in \mathcal{G}$.
Such a triple is said

irreducible, if G acts irreducibly on $\mathbb{V}$.
transitive, if G acts transitively on the unit sphere of $\mathbb{V}$.
symmetric, if $g(R) = R$, for all $g \in G$.

In 1962 James Simons succeed in giving a classification free proof of Berger theorem by defining the so-called holonomy systems. . This is triple

$$[\mathbb{V}, R, G]$$

where $\mathbb{V}$ is a Euclidean vector space, G is a Lie subgroup of $SO(\mathbb{V})$ and R is an algebraic Riemannian curvature tensor on $\mathbb{V}$ with values $R_{x,y} \in \mathcal{G}$.
Such a triple is said

irreducible, if G acts irreducibly on $\mathbb{V}$.
transitive, if G acts transitively on the unit sphere of $\mathbb{V}$.
symmetric, if $g(R) = R$, for all $g \in G$.

In 1962 James Simons succeed in giving a classification free proof of Berger theorem by defining the so-called holonomy systems. . This is triple

$$[\mathbb{V}, R, G]$$

where $\mathbb{V}$ is a Euclidean vector space, G is a Lie subgroup of $SO(\mathbb{V})$ and R is an algebraic Riemannian curvature tensor on $\mathbb{V}$ with values $R_{x,y} \in \mathcal{G}$.
Such a triple is said

irreducible, if G acts irreducibly on $\mathbb{V}$.
transitive, if G acts transitively on the unit sphere of $\mathbb{V}$.
symmetric, if $g(R) = R$, for all $g \in G$.

Euclidean submanifold geometry

A similar rôle as symmetric spaces play in Riemannian geometry is played by all the orbits of s-representations i.e., the isotropy representation of semisimple symmetric spaces.

For example the isotropy representation of the symmetric space $SL(n)/SO(n)$ can be regarded as $SO(n)$ acting by conjugation on the space of traceless symmetric matrices.

The isotropy representation of the Grassmannian $SO(n+k)/SO(n) \times SO(k)$ can be regarded as the action of $SO(n) \times SO(k)$ on $\mathbb{R}^{n \times k}$ given by

$$(g, h).A = gAh^{-1}$$

Euclidean submanifold geometry

A similar rôle as symmetric spaces play in Riemannian geometry is played by all the orbits of s-representations i.e., the isotropy representation of semisimple symmetric spaces.

For example the isotropy representation of the symmetric space $SL(n)/SO(n)$ can be regarded as $SO(n)$ acting by conjugation on the space of traceless symmetric matrices.

The isotropy representation of the Grassmannian $SO(n+k)/SO(n) \times SO(k)$ can be regarded as the action of $SO(n) \times SO(k)$ on $\mathbb{R}^{n \times k}$ given by

$$(g, h).A = gAh^{-1}$$

Euclidean submanifold geometry

A similar rôle as symmetric spaces play in Riemannian geometry is played by all the orbits of s-representations i.e., the isotropy representation of semisimple symmetric spaces.

For example the isotropy representation of the symmetric space $SL(n)/SO(n)$ can be regarded as $SO(n)$ acting by conjugation on the space of traceless symmetric matrices.

The isotropy representation of the Grassmannian $SO(n+k)/SO(n) \times SO(k)$ can be regarded as the action of $SO(n) \times SO(k)$ on $\mathbb{R}^{n\times k}$ given by

$$(g,h).A = gAh^{-1}$$

Introduction.

Riemannian holonomy.

Submanifold geometry and holonomy.

Holonomy systems and skew-torsion holonomy systems.

Applications to naturally reductive spaces.

Euclidean submanifold geometry

A similar rôle as symmetric spaces play in Riemannian geometry is played by all the orbits of s-representations i.e., the isotropy representation of semisimple symmetric spaces.

For example the isotropy representation of the symmetric space $SL(n)/SO(n)$ can be regarded as $SO(n)$ acting by conjugation on the space of traceless symmetric matrices.

The isotropy representation of the Grassmannian $SO(n+k)/SO(n) \times SO(k)$ can be regarded as the action of $SO(n) \times SO(k)$ on $\mathbb{R}^{n \times k}$ given by

$$(g, h).A = gAh^{-1}$$

A symmetric space is characterized by the property that the parallel transport τ_c along any curve c, starting at p is achieved by the differential of an isometry, i.e.,

$$dg_{|p} = \tau_c \qquad g \text{ is unique.} \qquad \text{(E. Cartan)}$$

On the other hand, an orbit $M = K.v \subset \mathbb{R}^N$ of an s-representation is characterized by a similar property, with respect to the normal connection. Namely, the normal parallel transport $\tau_c^{\perp}$, along any c in M, starting at p, can be achieved by the differential of some extrinsic isometry of M. That is, there exists an isometry of $\mathbb{R}^N$ such that $g(M) = M$ and

$$dg_{|\nu_p M} = \tau_c^{\perp} \quad g \text{ is } \underline{\text{not}}\text{, in general, unique.} \qquad \text{(O.-Sánchez)}$$

A symmetric space is characterized by the property that the parallel transport τ_c along any curve c, starting at p is achieved by the differential of an isometry, i.e.,

$$dg_{|p} = \tau_c \qquad g \text{ is unique.} \qquad \text{(E. Cartan)}$$

On the other hand, an orbit $M = K.v \subset \mathbb{R}^N$ of an s-representation is characterized by a similar property, with respect to the normal connection. Namely, the normal parallel transport $\tau_c^\perp$, along any c in M, starting at p, can be achieved by the differential of some extrinsic isometry of M. That is, there exists an isometry of $\mathbb{R}^N$ such that $g(M) = M$ and

$$dg_{|\nu_p M} = \tau_c^\perp \quad g \text{ is } \underline{\text{not}}\text{, in general, unique.} \qquad \text{(O.-Sánchez)}$$

A symmetric space is characterized by the property that the parallel transport τ_c along any curve c, starting at p is achieved by the differential of an isometry, i.e.,

$$dg_{|p} = \tau_c \qquad g \text{ is unique.} \qquad \text{(E. Cartan)}$$

On the other hand, an orbit $M = K.v \subset \mathbb{R}^N$ of an s-representation is characterized by a similar property, with respect to the normal connection. Namely, the normal parallel transport $\tau_c^{\perp}$, along any c in M, starting at p, can be achieved by the differential of some extrinsic isometry of M. That is, there exists an isometry of $\mathbb{R}^N$ such that $g(M) = M$ and

$$dg_{|\nu_p M} = \tau_c^{\perp} \quad g \text{ is } \underline{\text{not}}\text{, in general, unique.} \qquad \text{(O.-Sánchez)}$$

A symmetric space is characterized by the property that the parallel transport τ_c along any curve c, starting at p is achieved by the differential of an isometry, i.e.,

$$dg_{|p} = \tau_c \qquad\qquad g \text{ is unique.} \qquad\quad \text{(E. Cartan)}$$

On the other hand, an orbit $M = K.v \subset \mathbb{R}^N$ of an s-representation is characterized by a similar property, with respect to the normal connection. Namely, the normal parallel transport $\tau_c^\perp$, along any c in M, starting at p, can be achieved by the differential of some extrinsic isometry of M. That is, there exists an isometry of $\mathbb{R}^N$ such that $g(M) = M$ and

$$dg_{|\nu_p M} = \tau_c^\perp \qquad g \text{ is \underline{not}, in general, unique.} \qquad\quad \text{(O.-Sánchez)}$$

A symmetric space is characterized by the property that the parallel transport τ_c along any curve c, starting at p is achieved by the differential of an isometry, i.e.,

$$dg_{|p} = \tau_c \qquad g \text{ is unique.} \qquad \text{(E. Cartan)}$$

On the other hand, an orbit $M = K.v \subset \mathbb{R}^N$ of an s-representation is characterized by a similar property, with respect to the normal connection. Namely, the normal parallel transport $\tau_c^\perp$, along any c in M, starting at p, can be achieved by the differential of some extrinsic isometry of M. That is, there exists an isometry of $\mathbb{R}^N$ such that $g(M) = M$ and

$$dg_{|\nu_p M} = \tau_c^\perp \qquad g \text{ is } \underline{\text{not}}\text{, in general, unique.} \qquad \text{(O.-Sánchez)}$$

A symmetric space is characterized by the property that the parallel transport τ_c along any curve c, starting at p is achieved by the differential of an isometry, i.e.,

$$dg_{|p} = \tau_c \qquad g \text{ is unique.} \qquad \text{(E. Cartan)}$$

On the other hand, an orbit $M = K.v \subset \mathbb{R}^N$ of an s-representation is characterized by a similar property, with respect to the normal connection. Namely, the normal parallel transport $\tau_c^{\perp}$, along any c in M, starting at p, can be achieved by the differential of some extrinsic isometry of M. That is, there exists an isometry of $\mathbb{R}^N$ such that $g(M) = M$ and

$$dg_{|\nu_p M} = \tau_c^{\perp} \quad g \text{ is } \underline{\text{not}}\text{, in general, unique.} \qquad \text{(O.-Sánchez)}$$

A symmetric space is characterized by the property that the parallel transport τ_c along any curve c, starting at p is achieved by the differential of an isometry, i.e.,

$$dg_{|p} = \tau_c \qquad g \text{ is unique.} \qquad \text{(E. Cartan)}$$

On the other hand, an orbit $M = K.v \subset \mathbb{R}^N$ of an s-representation is characterized by a similar property, with respect to the normal connection. Namely, the normal parallel transport $\tau_c^\perp$, along any c in M, starting at p, can be achieved by the differential of some extrinsic isometry of M. That is, there exists an isometry of $\mathbb{R}^N$ such that $g(M) = M$ and

$$dg_{|\nu_p M} = \tau_c^\perp \quad g \text{ is \underline{not}, in general, unique.} \qquad \text{(O.-Sánchez)}$$

Informal Remark. The normal connection gives weaker information, in submanifold geometry, than the Levi-Civita connection in Riemannian Geometry.

Normal Holonomy Theorem(O.) *The normal holonomy group of a Euclidean submanifold, acts on the normal space, up to its fixed point set, as the isotropy representation of semisimple symmetric space.*

The holonomy of a symmetric space, without Euclidean factor, coincides with the isotropy (represented on the tangent space). So, the normal holonomy theorem can be phrased as follows: *the normal holonomy representation of a Euclidean submanifold coincides with the holonomy representation of a symmetric space.* This means that the normal holonomy representation coincides with a non-exceptional riemannian holonomy.

Informal Remark. The normal connection gives weaker information, in submanifold geometry, than the Levi-Civita connection in Riemannian Geometry.

Normal Holonomy Theorem(O.) *The normal holonomy group of a Euclidean submanifold, acts on the normal space, up to its fixed point set, as the isotropy representation of semisimple symmetric space.*

The holonomy of a symmetric space, without Euclidean factor, coincides with the isotropy (represented on the tangent space). So, the normal holonomy theorem can be phrased as follows: *the normal holonomy representation of a Euclidean submanifold coincides with the holonomy representation of a symmetric space*. This means that the normal holonomy representation coincides with a non-exceptional riemannian holonomy.

Informal Remark. The normal connection gives weaker information, in submanifold geometry, than the Levi-Civita connection in Riemannian Geometry.

Normal Holonomy Theorem(O.) *The normal holonomy group of a Euclidean submanifold, acts on the normal space, up to its fixed point set, as the isotropy representation of semisimple symmetric space.*

The holonomy of a symmetric space, without Euclidean factor, coincides with the isotropy (represented on the tangent space). So, the normal holonomy theorem can be phrased as follows: *the normal holonomy representation of a Euclidean submanifold coincides with the holonomy representation of a symmetric space.* This means that the normal holonomy representation coincides with a non-exceptional riemannian holonomy.

Introduction.

Riemannian holonomy.

Submanifold geometry and holonomy.

Holonomy systems and skew-torsion holonomy systems.

Applications to naturally reductive spaces.

Informal Remark. The normal connection gives weaker information, in submanifold geometry, than the Levi-Civita connection in Riemannian Geometry.

Normal Holonomy Theorem(O.) *The normal holonomy group of a Euclidean submanifold, acts on the normal space, up to its fixed point set, as the isotropy representation of semisimple symmetric space.*

The holonomy of a symmetric space, without Euclidean factor, coincides with the isotropy (represented on the tangent space). So, the normal holonomy theorem can be phrased as follows: *the normal holonomy representation of a Euclidean submanifold coincides with the holonomy representation of a symmetric space.* This means that the normal holonomy representation coincides with a non-exceptional riemannian holonomy.

Informal Remark. The normal connection gives weaker information, in submanifold geometry, than the Levi-Civita connection in Riemannian Geometry.

Normal Holonomy Theorem(O.) *The normal holonomy group of a Euclidean submanifold, acts on the normal space, up to its fixed point set, as the isotropy representation of semisimple symmetric space.*

The holonomy of a symmetric space, without Euclidean factor, coincides with the isotropy (represented on the tangent space). So, the normal holonomy theorem can be phrased as follows: *the normal holonomy representation of a Euclidean submanifold coincides with the holonomy representation of a symmetric space.* This means that the normal holonomy representation coincides with a non-exceptional riemannian holonomy.

Informal Remark. The normal connection gives weaker information, in submanifold geometry, than the Levi-Civita connection in Riemannian Geometry.

Normal Holonomy Theorem(O.) *The normal holonomy group of a Euclidean submanifold, acts on the normal space, up to its fixed point set, as the isotropy representation of semisimple symmetric space.*

The holonomy of a symmetric space, without Euclidean factor, coincides with the isotropy (represented on the tangent space). So, the normal holonomy theorem can be phrased as follows: *the normal holonomy representation of a Euclidean submanifold coincides with the holonomy representation of a symmetric space*. This means that the normal holonomy representation coincides with a non-exceptional riemannian holonomy.

Informal Remark. The normal connection gives weaker information, in submanifold geometry, than the Levi-Civita connection in Riemannian Geometry.

Normal Holonomy Theorem(O.) *The normal holonomy group of a Euclidean submanifold, acts on the normal space, up to its fixed point set, as the isotropy representation of semisimple symmetric space.*

The holonomy of a symmetric space, without Euclidean factor, coincides with the isotropy (represented on the tangent space). So, the normal holonomy theorem can be phrased as follows: *the normal holonomy representation of a Euclidean submanifold coincides with the holonomy representation of a symmetric space*. This means that the normal holonomy representation coincides with a non-exceptional riemannian holonomy.

Informal Remark. The normal connection gives weaker information, in submanifold geometry, than the Levi-Civita connection in Riemannian Geometry.

Normal Holonomy Theorem(O.) *The normal holonomy group of a Euclidean submanifold, acts on the normal space, up to its fixed point set, as the isotropy representation of semisimple symmetric space.*

The holonomy of a symmetric space, without Euclidean factor, coincides with the isotropy (represented on the tangent space). So, the normal holonomy theorem can be phrased as follows: *the normal holonomy representation of a Euclidean submanifold coincides with the holonomy representation of a symmetric space*. This means that the normal holonomy representation coincides with a non-exceptional riemannian holonomy.

The above theorem is not a *Berger-type theorem*, since it gives information only about the representation of the normal holonomy on the normal space, but not about the space (in this case the submanifold).

As we informally pointed out, the normal holonomy gives weaker information than the Riemannian holonomy. So, interesting applications of the normal holonomy, can only be given within a restrictive class of submanifolds, as, for instance, the following:

- *Homogeneous submanifolds.* ...
- *Submanifolds with constant principal curvatures.*
- *Complex submanifolds.*

In order to illustrate this, we ennounce two Berger-type theorems, for the last two classes of submanifolds.

The above theorem is not a *Berger-type theorem*, since it gives information only about the representation of the normal holonomy on the normal space, but not about the space (in this case the submanifold).

As we informally pointed out, the normal holonomy gives weaker information than the Riemannian holonomy. So, interesting applications of the normal holonomy, can only be given within a restrictive class of submanifolds, as, for instance, the following:

- *Homogeneous submanifolds.*
- *Submanifolds with constant principal curvatures.*
- *Complex submanifolds.*

In order to illustrate this, we ennounce two Berger-type theorems, for the last two classes of submanifolds.

The above theorem is not a *Berger-type theorem*, since it gives information only about the representation of the normal holonomy on the normal space, but not about the space (in this case the submanifold).

As we informally pointed out, the normal holonomy gives <u>weaker</u> information than the Riemannian holonomy. So, interesting applications of the normal holonomy, can only be given within a restrictive class of submanifolds, as, for instance, the following:

- *Homogeneous submanifolds.*
- *Submanifolds with constant principal curvatures.*
- *Complex submanifolds.*

In order to illustrate this, we ennounce two Berger-type theorems, for the last two classes of submanifolds.

The above theorem is not a *Berger-type theorem*, since it gives information only about the representation of the normal holonomy on the normal space, but not about the space (in this case the submanifold).

As we informally pointed out, the normal holonomy gives weaker information than the Riemannian holonomy. So, interesting applications of the normal holonomy, can only be given within a restrictive class of submanifolds, as, for instance, the following:

- *Homogeneous submanifolds.*
- *Submanifolds with constant principal curvatures.*
- *Complex submanifolds.*

In order to illustrate this, we ennounce two Berger-type theorems, for the last two classes of submanifolds.

The above theorem is not a *Berger-type theorem*, since it gives information only about the representation of the normal holonomy on the normal space, but not about the space (in this case the submanifold).

As we informally pointed out, the normal holonomy gives weaker information than the Riemannian holonomy. So, interesting applications of the normal holonomy, can only be given within a restrictive class of submanifolds, as, for instance, the following:

- *Homogeneous submanifolds.*
- *Submanifolds with constant principal curvatures.*
- *Complex submanifolds.*

In order to illustrate this, we ennounce two Berger-type theorems, for the last two classes of submanifolds.

The above theorem is not a *Berger-type theorem*, since it gives information only about the representation of the normal holonomy on the normal space, but not about the space (in this case the submanifold).

As we informally pointed out, the normal holonomy gives weaker information than the Riemannian holonomy. So, interesting applications of the normal holonomy, can only be given within a restrictive class of submanifolds, as, for instance, the following:

- *Homogeneous submanifolds.* • • •
- *Submanifolds with constant principal curvatures.*
- *Complex submanifolds.*

In order to illustrate this, we ennounce two Berger-type theorems, for the last two classes of submanifolds.

The above theorem is not a *Berger-type theorem*, since it gives information only about the representation of the normal holonomy on the normal space, but not about the space (in this case the submanifold).

As we informally pointed out, the normal holonomy gives weaker information than the Riemannian holonomy. So, interesting applications of the normal holonomy, can only be given within a restrictive class of submanifolds, as, for instance, the following:

- *Homogeneous submanifolds.*
- *Submanifolds with constant principal curvatures.*
- *Complex submanifolds.*

In order to illustrate this, we ennounce two Berger-type theorems, for the last two classes of submanifolds.

The first one is a reformulation of a remarkable result of G. Thorbergsson about the homogeneity of isoparametric submanifolds of higher codimension (which can also be proven by using normal holonomy methods).

Theorem (Thorbergsson). Let M be a submanifold of the sphere with constant principal curvatures. Assume that the normal holonomy group of M acts irreducibly and non-transitively. Then M is an orbit of an s-respresentation.

Theorem (Console, Di Scala, O.) Let M be a complete and full complex submanifold of $\mathbb{C}P^n$. If the normal holonomy of M is not transitive, then M is the (unique) complex orbit, in the projectivezed space, of an irreducible Hermitian s-representation (or equivalently, M is an extrinsic symmetric submanifold of $\mathbb{C}P^n$).

The first one is a reformulation of a remarkable result of G. Thorbergsson about the homogeneity of isoparametric submanifolds of higher codimension (which can also be proven by using normal holonomy methods).

Theorem (Thorbergsson). Let M be a submanifold of the sphere with constant principal curvatures. Assume that the normal holonomy group of M acts irreducibly and non-transitively. Then M is an orbit of an s-respresentation.

Theorem (Console, Di Scala, O.) Let M be a complete and full complex submanifold of $\mathbb{C}P^n$. If the normal holonomy of M is not transitive, then M is the (unique) complex orbit, in the projectivezed space, of an irreducible Hermitian s-representation (or equivalently, M is an extrinsic symmetric submanifold of $\mathbb{C}P^n$).

The first one is a reformulation of a remarkable result of G. Thorbergsson about the homogeneity of isoparametric submanifolds of higher codimension (which can also be proven by using normal holonomy methods).

Theorem (Thorbergsson). Let M be a submanifold of the sphere with constant principal curvatures. Assume that the normal holonomy group of M acts irreducibly and non-transitively. Then M is an orbit of an s-respresentation.

Theorem (Console, Di Scala, O.) Let M be a complete and full complex submanifold of $\mathbb{C}P^n$. If the normal holonomy of M is not transitive, then M is the (unique) complex orbit, in the projectivezed space, of an irreducible Hermitian s-representation (or equivalently, M is an extrinsic symmetric submanifold of $\mathbb{C}P^n$).

A central result, related to normal holonomy, is the so-called Rank Rigidity theorem for submanifolds.

Decompose the normal bundle of M as

$$\nu M = \nu_0 M \oplus \nu_0^{\perp} M$$

where $\nu_0 M$ is the maximal parallel and flat subbundle of νM.

Observe that the normal holonomy group acts on $\nu_0^{\perp} M$ as an s-representation.

$$\text{rank}\,(M) := \dim_M(\nu_0 M)$$

A central result, related to normal holonomy, is the so-called Rank Rigidity theorem for submanifolds.

Decompose the normal bundle of M as

$$\nu M = \nu_0 M \oplus \nu_0^{\perp} M$$

where $\nu_0 M$ is the maximal parallel and flat subbundle of νM.

Observe that the normal holonomy group acts on $\nu_0^{\perp} M$ as an s-representation.

$$\text{rank } (M) := \dim_M(\nu_0 M)$$

A central result, related to normal holonomy, is the so-called Rank Rigidity theorem for submanifolds.

Decompose the normal bundle of M as

$$\nu M = \nu_0 M \oplus \nu_0^{\perp} M$$

where $\nu_0 M$ is the maximal parallel and flat subbundle of νM.

Observe that the normal holonomy group acts on $\nu_0^{\perp} M$ as an s-representation.

$$\text{rank } (M) := \dim_M(\nu_0 M)$$

A central result, related to normal holonomy, is the so-called Rank Rigidity theorem for submanifolds.

Decompose the normal bundle of M as

$$\nu M = \nu_0 M \oplus \nu_0^{\perp} M$$

where $\nu_0 M$ is the maximal parallel and flat subbundle of νM.

Observe that the normal holonomy group acts on $\nu_0^{\perp} M$ as an s-representation.

$$\text{rank } (M) := \dim_M(\nu_0 M)$$

A central result, related to normal holonomy, is the so-called Rank Rigidity theorem for submanifolds.

Decompose the normal bundle of M as

$$\nu M = \nu_0 M \oplus \nu_0^{\perp} M$$

where $\nu_0 M$ is the maximal parallel and flat subbundle of νM.

Observe that the normal holonomy group acts on $\nu_0^{\perp} M$ as an s-representation.

$$\text{rank } (M) := \dim_M(\nu_0 M)$$

A central result, related to normal holonomy, is the so-called Rank Rigidity theorem for submanifolds.

Decompose the normal bundle of M as

$$\nu M = \nu_0 M \oplus \nu_0^{\perp} M$$

where $\nu_0 M$ is the maximal parallel and flat subbundle of νM.

Observe that the normal holonomy group acts on $\nu_0^{\perp} M$ as an s-representation.

$$\text{rank }(M) := \dim_M(\nu_0 M)$$

The rank rigidity theorem, in its full generality was proven by Di Scala and the author in 2004. At some step, though the result referes to Euclidean submanifolds, one needs to consider Riemannian submanifolds of Lorentzian space.

Let us only write the version for homogeneous submanifolds of the the rank rigidity theorem (1994).

The rank rigidity theorem, in its full generality was proven by Di Scala and the author in 2004. At some step, though the result referes to Euclidean submanifolds, one needs to consider Riemannian submanifolds of Lorentzian space.

Let us only write the version for homogeneous submanifolds of the the rank rigidity theorem (1994).

The rank rigidity theorem, in its full generality was proven by Di Scala and the author in 2004. At some step, though the result referes to Euclidean submanifolds, one needs to consider Riemannian submanifolds of Lorentzian space.

Let us only write the version for homogeneous submanifolds of the the rank rigidity theorem (1994).

The rank rigidity theorem, in its full generality was proven by Di Scala and the author in 2004. At some step, though the result referes to Euclidean submanifolds, one needs to consider Riemannian submanifolds of Lorentzian space.

Let us only write the version for homogeneous submanifolds of the the rank rigidity theorem (1994).

Theorem (O.) Let M^n, $n \geq 2$, be a full and irreducible homogeneous Euclidean submanifold. If rank $(M) \geq 2$, then M is an orbit of an (irreducilbe) s-representation.

The following corollary is a key fact for relating normal holonomy with tangent holonomy. More precisely, for giving a geometric proof of Berger holonomy theorem (O., Ann. of Math. 2005).

Corollary. Let $M^n = K.v$, $n \geq 2$, be a full an irreducible Euclidean homogeneous submanifold. Then any parallel normal field is K-invariant.

This implies the following

Corollary. Let $M^n = K.v$, , $n \geq 2$, be a full an irreducible Euclidean homogeneous submanifold. Then the projection to the normal space $\nu_v M$ of any Euclidean Killing field induced by K lies in the normal holonomy algebra.

Introduction.

Riemannian holonomy.

Submanifold geometry and holonomy.

Holonomy systems and skew-torsion holonomy systems.

Applications to naturally reductive spaces.

Theorem (O.) Let M^n, $n \geq 2$, be a full and irreducible homogeneous Euclidean submanifold. If rank $(M) \geq 2$, then M is an orbit of an (irreducilbe) s-representation.

The following corollary is a key fact for relating normal holonomy with tangent holonomy. More precisely, for giving a geometric proof of Berger holonomy theorem (O., Ann. of Math. 2005).

Corollary. Let $M^n = K.v$, $n \geq 2$, be a full an irreducible Euclidean homogeneous submanifold. Then any parallel normal field is K-invariant.

This implies the following

Corollary. Let $M^n = K.v$, , $n \geq 2$, be a full an irreducible Euclidean homogeneous submanifold. Then the projection to the normal space $\nu_v M$ of any Euclidean Killing field induced by K lies in the normal holonomy algebra.

Introduction.

Riemannian holonomy.

Submanifold geometry and holonomy.

Holonomy systems and skew-torsion holonomy systems.

Applications to naturally reductive spaces.

Theorem (O.) Let M^n, $n \geq 2$, be a full and irreducible homogeneous Euclidean submanifold. If rank $(M) \geq 2$, then M is an orbit of an (irreducilbe) s-representation.

The following corollary is a key fact for relating normal holonomy with tangent holonomy. More precisely, for giving a geometric proof of Berger holonomy theorem (O., Ann. of Math. 2005).

Corollary. Let $M^n = K.v$, $n \geq 2$, be a full an irreducible Euclidean homogeneous submanifold. Then any parallel normal field is K-invariant.

This implies the following

Corollary. Let $M^n = K.v$, , $n \geq 2$, be a full an irreducible Euclidean homogeneous submanifold. Then the projection to the normal space $\nu_v M$ of any Euclidean Killing field induced by K lies in the normal holonomy algebra.

Introduction.

Riemannian holonomy.

Submanifold geometry and holonomy.

Holonomy systems and skew-torsion holonomy systems.

Applications to naturally reductive spaces.

Theorem (O.) Let M^n, $n \geq 2$, be a full and irreducible homogeneous Euclidean submanifold. If rank $(M) \geq 2$, then M is an orbit of an (irreducilbe) s-representation.

The following corollary is a key fact for relating normal holonomy with tangent holonomy. More precisely, for giving a geometric proof of Berger holonomy theorem (O., Ann. of Math. 2005).

Corollary. Let $M^n = K.v$, $n \geq 2$, be a full an irreducible Euclidean homogeneous submanifold. Then any parallel normal field is K-invariant.

This implies the following

Corollary. Let $M^n = K.v$, , $n \geq 2$, be a full an irreducible Euclidean homogeneous submanifold. Then the projection to the normal space $\nu_v M$ of any Euclidean Killing field induced by K lies in the normal holonomy algebra.

Theorem (O.) Let M^n, $n \geq 2$, be a full and irreducible homogeneous Euclidean submanifold. If rank $(M) \geq 2$, then M is an orbit of an (irreducilbe) s-representation.

The following corollary is a key fact for relating normal holonomy with tangent holonomy. More precisely, for giving a geometric proof of Berger holonomy theorem (O., Ann. of Math. 2005).

Corollary. Let $M^n = K.v$, $n \geq 2$, be a full an irreducible Euclidean homogeneous submanifold. Then any parallel normal field is K-invariant.

This implies the following

Corollary. Let $M^n = K.v$, , $n \geq 2$, be a full an irreducible Euclidean homogeneous submanifold. Then the projection to the normal space $\nu_v M$ of any Euclidean Killing field induced by K lies in the normal holonomy algebra.

We recall, from Section 1, the definition of a holonomy system.

This is a triple,

$$[\mathbb{V}, R, G]$$

where $\mathbb{V}$ is a Euclidean vector space, G is a Lie subgroup of $SO(\mathbb{V})$ and R is an algebraic Riemannian curvature tensor on $\mathbb{V}$ with values $R_{x,y} \in \mathcal{G}$. Such a triple is said

irreducible, if G acts irreducibly on $\mathbb{V}$.
transitive, if G acts transitive on the unit sphere of $\mathbb{V}$.
symmetric, if $g(R) = R$, for all $g \in G$.

Simons proved the following Berger-type result which, as he pointed out, implies Berger holonomy theorem.

Simons Holonomy Theorem. An irreducible Riemannian holonomy system which is not transitive must be symmetric.

We recall, from Section 1, the definition of a holonomy system.

This is a triple,

$$[\mathbb{V}, R, G]$$

where $\mathbb{V}$ is a Euclidean vector space, G is a Lie subgroup of $SO(\mathbb{V})$ and R is an algebraic Riemannian curvature tensor on $\mathbb{V}$ with values $R_{x,y} \in \mathcal{G}$. Such a triple is said

irreducible, if G acts irreducibly on $\mathbb{V}$.
transitive, if G acts transitive on the unit sphere of $\mathbb{V}$.
symmetric, if $g(R) = R$, for all $g \in G$.

Simons proved the following Berger-type result which, as he pointed out, implies Berger holonomy theorem.

Simons Holonomy Theorem. An irreducible Riemannian holonomy system which is not transitive must be symmetric.

We recall, from Section 1, the definition of a holonomy system.

This is a triple,

$$[\mathbb{V}, R, G]$$

where $\mathbb{V}$ is a Euclidean vector space, G is a Lie subgroup of $SO(\mathbb{V})$ and R is an algebraic Riemannian curvature tensor on $\mathbb{V}$ with values $R_{x,y} \in \mathcal{G}$. Such a triple is said

irreducible, if G acts irreducibly on $\mathbb{V}$.
transitive, if G acts transitive on the unit sphere of $\mathbb{V}$.
symmetric, if $g(R) = R$, for all $g \in G$.

Simons proved the following Berger-type result which, as he pointed out, implies Berger holonomy theorem.

Simons Holonomy Theorem. An irreducible Riemannian holonomy system which is not transitive must be symmetric.

We recall, from Section 1, the definition of a holonomy system.

This is a triple,

$$[\mathbb{V}, R, G]$$

where $\mathbb{V}$ is a Euclidean vector space, G is a Lie subgroup of $SO(\mathbb{V})$ and R is an algebraic Riemannian curvature tensor on $\mathbb{V}$ with values $R_{x,y} \in \mathcal{G}$. Such a triple is said

irreducible, if G acts irreducibly on $\mathbb{V}$.
transitive, if G acts transitive on the unit sphere of $\mathbb{V}$.
symmetric, if $g(R) = R$, for all $g \in G$.

Simons proved the following Berger-type result which, as he pointed out, implies Berger holonomy theorem.

Simons Holonomy Theorem. An irreducible Riemannian holonomy system which is not transitive must be symmetric.

We recall, from Section 1, the definition of a holonomy system.

This is a triple,

$$[\mathbb{V}, R, G]$$

where $\mathbb{V}$ is a Euclidean vector space, G is a Lie subgroup of $SO(\mathbb{V})$ and R is an algebraic Riemannian curvature tensor on $\mathbb{V}$ with values $R_{x,y} \in \mathcal{G}$. Such a triple is said

irreducible, if G acts irreducibly on $\mathbb{V}$.
transitive, if G acts transitive on the unit sphere of $\mathbb{V}$.
symmetric, if $g(R) = R$, for all $g \in G$.

Simons proved the following Berger-type result which, as he pointed out, implies Berger holonomy theorem.

Simons Holonomy Theorem. An irreducible Riemannian holonomy system which is not transitive must be symmetric.

We recall, from Section 1, the definition of a holonomy system.

This is a triple,

$$[\mathbb{V}, R, G]$$

where $\mathbb{V}$ is a Euclidean vector space, G is a Lie subgroup of $SO(\mathbb{V})$ and R is an algebraic Riemannian curvature tensor on $\mathbb{V}$ with values $R_{x,y} \in \mathcal{G}$. Such a triple is said

irreducible, if G acts irreducibly on $\mathbb{V}$.
transitive, if G acts transitive on the unit sphere of $\mathbb{V}$.
symmetric, if $g(R) = R$, for all $g \in G$.

Simons proved the following Berger-type result which, as he pointed out, implies Berger holonomy theorem.

Simons Holonomy Theorem. An irreducible Riemannian holonomy system which is not transitive must be symmetric.

We recall, from Section 1, the definition of a holonomy system.

This is a triple,

$$[\mathbb{V}, R, G]$$

where $\mathbb{V}$ is a Euclidean vector space, G is a Lie subgroup of $SO(\mathbb{V})$ and R is an algebraic Riemannian curvature tensor on $\mathbb{V}$ with values $R_{x,y} \in \mathcal{G}$. Such a triple is said

irreducible, if G acts irreducibly on $\mathbb{V}$.
transitive, if G acts transitive on the unit sphere of $\mathbb{V}$.
symmetric, if $g(R) = R$, for all $g \in G$.

Simons proved the following Berger-type result which, as he pointed out, implies Berger holonomy theorem.

Simons Holonomy Theorem. An irreducible Riemannian holonomy system which is not transitive must be symmetric.

We recall, from Section 1, the definition of a holonomy system.

This is a triple,

$$[\mathbb{V}, R, G]$$

where $\mathbb{V}$ is a Euclidean vector space, G is a Lie subgroup of $SO(\mathbb{V})$ and R is an algebraic Riemannian curvature tensor on $\mathbb{V}$ with values $R_{x,y} \in \mathcal{G}$. Such a triple is said

irreducible, if G acts irreducibly on $\mathbb{V}$.
transitive, if G acts transitive on the unit sphere of $\mathbb{V}$.
symmetric, if $g(R) = R$, for all $g \in G$.

Simons proved the following Berger-type result which, as he pointed out, implies Berger holonomy theorem.

Simons Holonomy Theorem. An irreducible Riemannian holonomy system which is not transitive must be symmetric.

We recall, from Section 1, the definition of a holonomy system.

This is a triple,

$$[\mathbb{V}, R, G]$$

where $\mathbb{V}$ is a Euclidean vector space, G is a Lie subgroup of $SO(\mathbb{V})$ and R is an algebraic Riemannian curvature tensor on $\mathbb{V}$ with values $R_{x,y} \in \mathcal{G}$. Such a triple is said

irreducible, if G acts irreducibly on $\mathbb{V}$.
transitive, if G acts transitive on the unit sphere of $\mathbb{V}$.
symmetric, if $g(R) = R$, for all $g \in G$.

Simons proved the following Berger-type result which, as he pointed out, implies Berger holonomy theorem.

Simons Holonomy Theorem. An irreducible Riemannian holonomy system which is not transitive must be symmetric.

We recall, from Section 1, the definition of a holonomy system.

This is a triple,

$$[\mathbb{V}, R, G]$$

where $\mathbb{V}$ is a Euclidean vector space, G is a Lie subgroup of $SO(\mathbb{V})$ and R is an algebraic Riemannian curvature tensor on $\mathbb{V}$ with values $R_{x,y} \in \mathcal{G}$. Such a triple is said

irreducible, if G acts irreducibly on $\mathbb{V}$.
transitive, if G acts transitive on the unit sphere of $\mathbb{V}$.
symmetric, if $g(R) = R$, for all $g \in G$.

Simons proved the following Berger-type result which, as he pointed out, implies Berger holonomy theorem.

Simons Holonomy Theorem. An irreducible Riemannian holonomy system which is not transitive must be symmetric.

We recall, from Section 1, the definition of a holonomy system.

This is a triple,

$$[\mathbb{V}, R, G]$$

where $\mathbb{V}$ is a Euclidean vector space, G is a Lie subgroup of $SO(\mathbb{V})$ and R is an algebraic Riemannian curvature tensor on $\mathbb{V}$ with values $R_{x,y} \in \mathcal{G}$. Such a triple is said

irreducible, if G acts irreducibly on $\mathbb{V}$.
transitive, if G acts transitive on the unit sphere of $\mathbb{V}$.
symmetric, if $g(R) = R$, for all $g \in G$.

Simons proved the following Berger-type result which, as he pointed out, implies Berger holonomy theorem.

Simons Holonomy Theorem. An irreducible Riemannian holonomy system which is not transitive must be symmetric.

A general holonomy systems is the direct product of irreducible ones, provided one replaces the group G by a normal subgroup G^0 with the same orbits (where $\mathcal{G}^0$ is the linear span of the curvatures endomorphisms)

The proof given by Simons, though elemental, is algebraic and involved. But one can prove Simons theorem by using one of the results we have commented before. Namely,

Corollary. Let $M^n = K.v$, , $n \geq 2$, be a full an irreducible Euclidean homogeneous submanifold. Then the projection to the normal space $\nu_v M$ of any Euclidean Killing field induced by K lies in the normal holonomy algebra.

The above corollary, combined with very simple observations, implies Simons holonomy theorem (O., L'Enseignement Mathématique 2005)

A general holonomy systems is the direct product of irreducible ones, provided one replaces the group G by a normal subgroup G^0 with the same orbits (where $\mathcal{G}^0$ is the linear span of the curvatures endomorphisms)

The proof given by Simons, though elemental, is algebraic and involved. But one can prove Simons theorem by using one of the results we have commented before. Namely,

Corollary. Let $M^n = K.v$, , $n \geq 2$, be a full an irreducible Euclidean homogeneous submanifold. Then the projection to the normal space $\nu_v M$ of any Euclidean Killing field induced by K lies in the normal holonomy algebra.

The above corollary, combined with very simple observations, implies Simons holonomy theorem (O., L'Enseignement Mathématique 2005)

A general holonomy systems is the direct product of irreducible ones, provided one replaces the group G by a normal subgroup G^0 with the same orbits (where $\mathcal{G}^0$ is the linear span of the curvatures endomorphisms)

The proof given by Simons, though elemental, is algebraic and involved. But one can prove Simons theorem by using one of the results we have commented before. Namely,

Corollary. Let $M^n = K.v$, , $n \geq 2$, be a full an irreducible Euclidean homogeneous submanifold. Then the projection to the normal space $\nu_v M$ of any Euclidean Killing field induced by K lies in the normal holonomy algebra.

The above corollary, combined with very simple observations, implies Simons holonomy theorem (O., L'Enseignement Mathématique 2005)

Introduction.

Riemannian holonomy.

Submanifold geometry and holonomy.

Holonomy systems and skew-torsion holonomy systems.

Applications to naturally reductive spaces.

A general holonomy systems is the direct product of irreducible ones, provided one replaces the group G by a normal subgroup G^0 with the same orbits (where $\mathcal{G}^0$ is the linear span of the curvatures endomorphisms)

The proof given by Simons, though elemental, is algebraic and involved. But one can prove Simons theorem by using one of the results we have commented before. Namely,

Corollary. Let $M^n = K.v$, , $n \geq 2$, be a full an irreducible Euclidean homogeneous submanifold. Then the projection to the normal space $\nu_v M$ of any Euclidean Killing field induced by K lies in the normal holonomy algebra.

The above corollary, combined with very simple observations, implies Simons holonomy theorem (O., L'Enseignement Mathématique 2005)

A general holonomy systems is the direct product of irreducible ones, provided one replaces the group G by a normal subgroup G^0 with the same orbits (where $\mathcal{G}^0$ is the linear span of the curvatures endomorphisms)

The proof given by Simons, though elemental, is algebraic and involved. But one can prove Simons theorem by using one of the results we have commented before. Namely,

Corollary. Let $M^n = K.v$, , $n \geq 2$, be a full an irreducible Euclidean homogeneous submanifold. Then the projection to the normal space $\nu_v M$ of any Euclidean Killing field induced by K lies in the normal holonomy algebra. • • •

The above corollary, combined with very simple observations, implies Simons holonomy theorem (O., L'Enseignement Mathématique 2005)

Let us replace, in the definition of a holonomy system, the algebraic Riemannian curvature tensor R by a 1-form θ with values in the Lie algebra $\mathcal{G}$ and such that $\langle \theta_X Y, Z \rangle$ is totally skew.

In this case the triple $[\mathbb{V}, \theta, G]$ is called a skew-torsion holonomy system.

Such a $\mathcal{G}$-valued 1-form arises, usually, as the difference tensor between two different metric connections with the same geodesics. The so-called connections with skew-torsion (when one of the connections is the Levi-Civita connection). We will refer later to this.

Let us replace, in the definition of a holonomy system, the algebraic Riemannian curvature tensor R by a 1-form θ with values in the Lie algebra $\mathcal{G}$ and such that $\langle \theta_X Y, Z \rangle$ is totally skew.

In this case the triple $[\mathbb{V}, \theta, G]$ is called a skew-torsion holonomy system.

Such a $\mathcal{G}$-valued 1-form arises, usually, as the difference tensor between two different metric connections with the same geodesics. The so-called connections with skew-torsion (when one of the connections is the Levi-Civita connection). We will refer later to this.

Let us replace, in the definition of a holonomy system, the algebraic Riemannian curvature tensor R by a 1-form θ with values in the Lie algebra $\mathcal{G}$ and such that $\langle \theta_X Y, Z \rangle$ is totally skew.

In this case the triple $[\mathbb{V}, \theta, G]$ is called a skew-torsion holonomy system.

Such a $\mathcal{G}$-valued 1-form arises, usually, as the difference tensor between two different metric connections with the same geodesics. The so-called connections with skew-torsion (when one of the connections is the Levi-Civita connection). We will refer later to this.

Let us replace, in the definition of a holonomy system, the algebraic Riemannian curvature tensor R by a 1-form θ with values in the Lie algebra $\mathcal{G}$ and such that $\langle \theta_X Y, Z \rangle$ is totally skew.

In this case the triple $[\mathbb{V}, \theta, G]$ is called a skew-torsion holonomy system.

Such a $\mathcal{G}$-valued 1-form arises, usually, as the difference tensor between two different metric connections with the same geodesics. The so-called connections with skew-torsion (when one of the connections is the Levi-Civita connection). We will refer later to this.

Introduction.

Riemannian holonomy.

Submanifold geometry and holonomy.

Holonomy systems and skew-torsion holonomy systems.

Applications to naturally reductive spaces.

Let us replace, in the definition of a holonomy system, the algebraic Riemannian curvature tensor R by a 1-form θ with values in the Lie algebra $\mathcal{G}$ and such that $\langle \theta_X Y, Z \rangle$ is totally skew.

In this case the triple $[\mathbb{V}, \theta, G]$ is called a skew-torsion holonomy system.

Such a $\mathcal{G}$-valued 1-form arises, usually, as the difference tensor between two different metric connections with the same geodesics. The so-called connections with skew-torsion (when one of the connections is the Levi-Civita connection). We will refer later to this.

Introduction.

Riemannian holonomy.

Submanifold geometry and holonomy.

Holonomy systems and skew-torsion holonomy systems.

Applications to naturally reductive spaces.

Let us replace, in the definition of a holonomy system, the algebraic Riemannian curvature tensor R by a 1-form θ with values in the Lie algebra $\mathcal{G}$ and such that $\langle \theta_X Y, Z \rangle$ is totally skew.

In this case the triple $[\mathbb{V}, \theta, G]$ is called a skew-torsion holonomy system.

Such a $\mathcal{G}$-valued 1-form arises, usually, as the difference tensor between two different metric connections with the same geodesics. The so-called connections with skew-torsion (when one of the connections is the Levi-Civita connection).
We will refer later to this.

Let us replace, in the definition of a holonomy system, the algebraic Riemannian curvature tensor R by a 1-form θ with values in the Lie algebra $\mathcal{G}$ and such that $\langle \theta_X Y, Z \rangle$ is totally skew.

In this case the triple $[\mathbb{V}, \theta, G]$ is called a skew-torsion holonomy system.

Such a $\mathcal{G}$-valued 1-form arises, usually, as the difference tensor between two different metric connections with the same geodesics. The so-called connections with skew-torsion (when one of the connections is the Levi-Civita connection). We will refer later to this.

Let us replace, in the definition of a holonomy system, the algebraic Riemannian curvature tensor R by a 1-form θ with values in the Lie algebra $\mathcal{G}$ and such that $\langle \theta_X Y, Z \rangle$ is totally skew.

In this case the triple $[\mathbb{V}, \theta, G]$ is called a skew-torsion holonomy system.

Such a $\mathcal{G}$-valued 1-form arises, usually, as the difference tensor between two different metric connections with the same geodesics. The so-called connections with skew-torsion (when one of the connections is the Levi-Civita connection). We will refer later to this.

Decomposition of skew-torsion holonomy systems: Let G' be the subgroup of G with Lie algebra

$$\mathcal{G}' = \{g(\theta)_x : g \in G,\ x \in \mathbb{V}\}$$

One has, as it is standard to prove,

$$\mathbb{V} = \mathbb{V}_0 \oplus \cdots \oplus \mathbb{V}_k \qquad \text{(orthogonally)}$$

$$G' = G'_1 \times \cdots \times G'_k$$

where G'_i acts irreducibly on $\mathbb{V}_i$ and trivially on $\mathbb{V}_j$, $i \neq j$ (Agricola-Friedrich).

Decomposition of skew-torsion holonomy systems: Let G' be the subgroup of G with Lie algebra

$$\mathcal{G}' = \{g(\theta)_x : g \in G, \ x \in \mathbb{V}\}$$

One has, as it is standard to prove,

$$\mathbb{V} = \mathbb{V}_0 \oplus \cdots \oplus \mathbb{V}_k \qquad \text{(orthogonally)}$$

$$G' = G'_1 \times \cdots \times G'_k$$

where G'_i acts irreducibly on $\mathbb{V}_i$ and trivially on $\mathbb{V}_j$, $i \neq j$ (Agricola-Friedrich).

Decomposition of skew-torsion holonomy systems: Let G' be the subgroup of G with Lie algebra

$$\mathcal{G}' = \{g(\theta)_x : g \in G,\ x \in \mathbb{V}\}$$

One has, as it is standard to prove,

$$\mathbb{V} = \mathbb{V}_0 \oplus \cdots \oplus \mathbb{V}_k \qquad \text{(orthogonally)}$$

$$G' = G'_1 \times \cdots \times G'_k$$

where G'_i acts irreducibly on $\mathbb{V}_i$ and trivially on $\mathbb{V}_j$, $i \neq j$ (Agricola-Friedrich).

Decomposition of skew-torsion holonomy systems: Let G' be the subgroup of G with Lie algebra

$$\mathcal{G}' = \{g(\theta)_x : g \in G,\ x \in \mathbb{V}\}$$

One has, as it is standard to prove,

$$\mathbb{V} = \mathbb{V}_0 \oplus \cdots \oplus \mathbb{V}_k \qquad \text{(orthogonally)}$$

$$G' = G'_1 \times \cdots \times G'_k$$

where G'_i acts irreducibly on $\mathbb{V}_i$ and trivially on $\mathbb{V}_j$, $i \neq j$ (Agricola-Friedrich).

Decomposition of skew-torsion holonomy systems: Let G' be the subgroup of G with Lie algebra

$$\mathcal{G}' = \{g(\theta)_x : g \in G,\ x \in \mathbb{V}\}$$

One has, as it is standard to prove,

$$\mathbb{V} = \mathbb{V}_0 \oplus \cdots \oplus \mathbb{V}_k \qquad \text{(orthogonally)}$$

$$G' = G'_1 \times \cdots \times G'_k$$

where G'_i acts irreducibly on $\mathbb{V}_i$ and trivially on $\mathbb{V}_j$, $i \neq j$ (Agricola-Friedrich).

Decomposition of skew-torsion holonomy systems: Let G' be the subgroup of G with Lie algebra

$$\mathcal{G}' = \{g(\theta)_x : g \in G,\ x \in \mathbb{V}\}$$

One has, as it is standard to prove,

$$\mathbb{V} = \mathbb{V}_0 \oplus \cdots \oplus \mathbb{V}_k \qquad \text{(orthogonally)}$$

$$G' = G'_1 \times \cdots \times G'_k$$

where G'_i acts irreducibly on $\mathbb{V}_i$ and trivially on $\mathbb{V}_j$, $i \neq j$ (Agricola-Friedrich).

Moreover, if $X \in \mathcal{SO}(\mathbb{V}_i)$, then

$$[X, \mathcal{G}_i'] = 0 \quad \Rightarrow \quad X = 0.$$

or, equivalently, G_i' is not Hermitian. (In particular, G_i is semisimple; Agricola-Friedrich).

This implies that

$$G = G_0 \times G' = G_0 \times G_1' \times \cdots \times G_k'$$

where G_0 acts only on $\mathbb{V}_0$ (no more information about G_0, which can be arbitrary).

Last equality is not true for holonomy systems.

Introduction.

Riemannian holonomy.

Submanifold geometry and holonomy.

Holonomy systems and skew-torsion holonomy systems.

Applications to naturally reductive spaces.

Moreover, if $X \in \mathcal{SO}(\mathbb{V}_i)$, then

$$[X, \mathcal{G}_i'] = 0 \quad \Rightarrow \quad X = 0.$$

or, equivalently, G_i' is not Hermitian. (In particular, G_i is semisimple; Agricola-Friedrich).

This implies that

$$G = G_0 \times G' = G_0 \times G_1' \times \cdots \times G_k'$$

where G_0 acts only on $\mathbb{V}_0$ (no more information about G_0, which can be arbitrary).

Last equality is not true for holonomy systems.

Moreover, if $X \in \mathcal{SO}(\mathbb{V}_i)$, then

$$[X, \mathcal{G}_i'] = 0 \quad \Rightarrow \quad X = 0.$$

or, equivalently, G_i' is not Hermitian. (In particular, G_i is semisimple; Agricola-Friedrich).

This implies that

$$G = G_0 \times G' = G_0 \times G_1' \times \cdots \times G_k'$$

where G_0 acts only on $\mathbb{V}_0$ (no more information about G_0, which can be arbitrary).

Last equality is not true for holonomy systems.

Moreover, if $X \in \mathcal{SO}(\mathbb{V}_i)$, then

$$[X, \mathcal{G}_i'] = 0 \quad \Rightarrow \quad X = 0.$$

or, equivalently, G_i' is not Hermitian. (In particular, G_i is semisimple; Agricola-Friedrich).

This implies that

$$G = G_0 \times G' = G_0 \times G_1' \times \cdots \times G_k'$$

where G_0 acts only on $\mathbb{V}_0$ (no more information about G_0, which can be arbitrary).

Last equality is not true for holonomy systems.

Moreover, if $X \in \mathcal{SO}(\mathbb{V}_i)$, then

$$[X, \mathcal{G}_i'] = 0 \quad \Rightarrow \quad X = 0.$$

or, equivalently, G_i' is not Hermitian. (In particular, G_i is semisimple; Agricola-Friedrich).

This implies that

$$G = G_0 \times G' = G_0 \times G_1' \times \cdots \times G_k'$$

where G_0 acts only on $\mathbb{V}_0$ (no more information about G_0, which can be arbitrary).

Last equality is not true for holonomy systems.

Moreover, if $X \in \mathcal{SO}(\mathbb{V}_i)$, then

$$[X, \mathcal{G}_i'] = 0 \quad \Rightarrow \quad X = 0.$$

or, equivalently, G_i' is not Hermitian. (In particular, G_i is semisimple; Agricola-Friedrich).

This implies that

$$G = G_0 \times G' = G_0 \times G_1' \times \cdots \times G_k'$$

where G_0 acts only on $\mathbb{V}_0$ (no more information about G_0, which can be arbitrary).

Last equality is not true for holonomy systems.

Introduction.

Riemannian holonomy.

Submanifold geometry and holonomy.

Holonomy systems and skew-torsion holonomy systems.

Applications to naturally reductive spaces.

Moreover, if $X \in \mathcal{SO}(\mathbb{V}_i)$, then

$$[X, \mathcal{G}_i'] = 0 \quad \Rightarrow \quad X = 0.$$

or, equivalently, G_i' is not Hermitian. (In particular, G_i is semisimple; Agricola-Friedrich).

This implies that

$$G = G_0 \times G' = G_0 \times G_1' \times \cdots \times G_k'$$

where G_0 acts only on $\mathbb{V}_0$ (no more information about G_0, which can be arbitrary).

Last equality is not true for holonomy systems.

Moreover, if $X \in \mathcal{SO}(\mathbb{V}_i)$, then

$$[X, \mathcal{G}_i'] = 0 \quad \Rightarrow \quad X = 0.$$

or, equivalently, G_i' is not Hermitian. (In particular, G_i is semisimple; Agricola-Friedrich).

This implies that

$$G = G_0 \times G' = G_0 \times G_1' \times \cdots \times G_k'$$

where G_0 acts only on $\mathbb{V}_0$ (no more information about G_0, which can be arbitrary).

Last equality is not true for holonomy systems.

The same proof as that given for Simons holonomy theorem, that we commented before, applies to prove a similar result for skew-torsion holonomy systems. Namely,

An irreducible non-transitive skew-torsion holonomy system must be symmetric.

Unlike the case of holonomy systems, no transitive groups can occur, except the full orthogonal group. In fact, the transitive cases were essentially disregarded, case by case, by Agricola-Friedrich (Math. Ann., 2004). Let us see our last Berger-type theorem. So,

The same proof as that given for Simons holonomy theorem, that we commented before, applies to prove a similar result for skew-torsion holonomy systems. Namely,

An irreducible non-transitive skew-torsion holonomy system must be symmetric.

Unlike the case of holonomy systems, no transitive groups can occur, except the full orthogonal group. In fact, the transitive cases were essentially disregarded, case by case, by Agricola-Friedrich (Math. Ann., 2004). Let us see our last Berger-type theorem. So,

Introduction.

Riemannian holonomy.

Submanifold geometry and holonomy.

Holonomy systems and skew-torsion holonomy systems.

Applications to naturally reductive spaces.

The same proof as that given for Simons holonomy theorem, that we commented before, applies to prove a similar result for skew-torsion holonomy systems. Namely,

An irreducible non-transitive skew-torsion holonomy system must be symmetric.

Unlike the case of holonomy systems, no transitive groups can occur, except the full orthogonal group. In fact, the transitive cases were essentially disregarded, case by case, by Agricola-Friedrich (Math. Ann., 2004). Let us see our last Berger-type theorem. So,

The same proof as that given for Simons holonomy theorem, that we commented before, applies to prove a similar result for skew-torsion holonomy systems. Namely,

An irreducible non-transitive skew-torsion holonomy system must be symmetric.

Unlike the case of holonomy systems, no transitive groups can occur, except the full orthogonal group. In fact, the transitive cases were essentially disregarded, case by case, by Agricola-Friedrich (Math. Ann., 2004). Let us see our last Berger-type theorem. So,

Skew-torsion Holonomy Theorem (Nagy; O.-Reggiani). *Let* $[\mathbb{V}, \Theta, G]$, $\Theta \neq 0$, *be an irreducible skew-torsion holonomy system with* $G \neq \mathrm{SO}(\mathbb{V})$. *Then* $[\mathbb{V}, \Theta, G]$ *is symmetric and non-transitive.* *Moreover,*

1. $(\mathbb{V}, [\ ,\ ])$ *is an orthogonal simple Lie algebra, of rank at least 2, with respect to the bracket* $[x, y] = \Theta_x y$;
2. $G = \mathrm{Ad}(H)$, *where* H *is the connected Lie group associated to the Lie algebra* $(\mathbb{V}, [\ ,\ ])$;
3. Θ *is unique, up to a scalar multiple.*

Skew-torsion Holonomy Theorem (Nagy; O.-Reggiani). *Let* $[\mathbb{V}, \Theta, G]$, $\Theta \neq 0$, *be an irreducible skew-torsion holonomy system with* $G \neq \mathrm{SO}(\mathbb{V})$. *Then* $[\mathbb{V}, \Theta, G]$ *is symmetric and non-transitive. Moreover,*

1. $(\mathbb{V}, [\ ,\ ])$ *is an orthogonal simple Lie algebra, of rank at least 2, with respect to the bracket* $[x, y] = \Theta_x y$;
2. $G = \mathrm{Ad}(H)$, *where* H *is the connected Lie group associated to the Lie algebra* $(\mathbb{V}, [\ ,\ ])$;
3. Θ *is unique, up to a scalar multiple.*

Skew-torsion Holonomy Theorem (Nagy; O.-Reggiani). *Let* $[\mathbb{V}, \Theta, G]$, $\Theta \neq 0$, *be an irreducible skew-torsion holonomy system with* $G \neq \mathrm{SO}(\mathbb{V})$. *Then* $[\mathbb{V}, \Theta, G]$ *is symmetric and non-transitive. Moreover,*

1. $(\mathbb{V}, [\ ,\ ])$ *is an orthogonal simple Lie algebra, of rank at least* 2, *with respect to the bracket* $[x, y] = \Theta_x y$;
2. $G = \mathrm{Ad}(H)$, *where* H *is the connected Lie group associated to the Lie algebra* $(\mathbb{V}, [\ ,\ ])$;
3. Θ *is unique, up to a scalar multiple.*

Skew-torsion Holonomy Theorem (Nagy; O.-Reggiani). *Let* $[\mathbb{V}, \Theta, G]$, $\Theta \neq 0$, *be an irreducible skew-torsion holonomy system with* $G \neq \mathrm{SO}(\mathbb{V})$. *Then* $[\mathbb{V}, \Theta, G]$ *is symmetric and non-transitive. Moreover,*

1. $(\mathbb{V}, [\ ,\ ])$ *is an orthogonal simple Lie algebra, of rank at least* 2, *with respect to the bracket* $[x, y] = \Theta_x y$;
2. $G = \mathrm{Ad}(H)$, *where* H *is the connected Lie group associated to the Lie algebra* $(\mathbb{V}, [\ ,\ ])$;
3. Θ *is unique, up to a scalar multiple.*

Skew-torsion Holonomy Theorem (Nagy; O.-Reggiani). *Let* $[\mathbb{V}, \Theta, G]$, $\Theta \neq 0$, *be an irreducible skew-torsion holonomy system with* $G \neq \mathrm{SO}(\mathbb{V})$. *Then* $[\mathbb{V}, \Theta, G]$ *is symmetric and non-transitive. Moreover,*

1. $(\mathbb{V}, [\ ,\ ])$ *is an orthogonal simple Lie algebra, of rank at least* 2, *with respect to the bracket* $[x, y] = \Theta_x y$;
2. $G = \mathrm{Ad}(H)$, *where* H *is the connected Lie group associated to the Lie algebra* $(\mathbb{V}, [\ ,\ ])$;
3. Θ *is unique, up to a scalar multiple.*

Skew-torsion Holonomy Theorem (Nagy; O.-Reggiani). *Let* $[\mathbb{V}, \Theta, G]$, $\Theta \neq 0$, *be an irreducible skew-torsion holonomy system with* $G \neq \mathrm{SO}(\mathbb{V})$. *Then* $[\mathbb{V}, \Theta, G]$ *is symmetric and non-transitive. Moreover,*

1. $(\mathbb{V}, [\ ,\ ])$ *is an orthogonal simple Lie algebra, of rank at least* 2, *with respect to the bracket* $[x, y] = \Theta_x y$;
2. $G = \mathrm{Ad}(H)$, *where* H *is the connected Lie group associated to the Lie algebra* $(\mathbb{V}, [\ ,\ ])$;
3. Θ *is unique, up to a scalar multiple.*

This theorem was independently proved by Paul-A. Nagy by using the classification of the so-called Berger algebras, though in this final form a little earlier.

Our approach is geometric, based on submanifold geometry, and does not use any classification result. Our motivation for finding such a result came from homogenous spaces. In particular for explaining the inextendibility of the presentation group of an isotropy irreducible space which is not the sphere.

In the final part of this talk we will show these applications. First we will do some remarks for motivating why the only transitive group is the full orthogonal group.

This theorem was independently proved by Paul-A. Nagy by using the classification of the so-called Berger algebras, though in this final form a little earlier.

Our approach is geometric, based on submanifold geometry, and does not use any classification result. Our motivation for finding such a result came from homogenous spaces. In particular for explaining the inextendibility of the presentation group of an isotropy irreducible space which is not the sphere.

In the final part of this talk we will show these applications. First we will do some remarks for motivating why the only transitive group is the full orthogonal group.

This theorem was independently proved by Paul-A. Nagy by using the classification of the so-called Berger algebras, though in this final form a little earlier.

Our approach is geometric, based on submanifold geometry, and does not use any classification result. Our motivation for finding such a result came from homogenous spaces. In particular for explaining the inextendibility of the presentation group of an isotropy irreducible space which is not the sphere.

In the final part of this talk we will show these applications. First we will do some remarks for motivating why the only transitive group is the full orthogonal group.

This theorem was independently proved by Paul-A. Nagy by using the classification of the so-called Berger algebras, though in this final form a little earlier.

Our approach is geometric, based on submanifold geometry, and does not use any classification result. Our motivation for finding such a result came from homogenous spaces. In particular for explaining the inextendibility of the presentation group of an isotropy irreducible space which is not the sphere.

In the final part of this talk we will show these applications. First we will do some remarks for motivating why the only transitive group is the full orthogonal group.

Introduction.

Riemannian holonomy.

Submanifold geometry and holonomy.

Holonomy systems and skew-torsion holonomy systems.

Applications to naturally reductive spaces.

This theorem was independently proved by Paul-A. Nagy by using the classification of the so-called Berger algebras, though in this final form a little earlier.

Our approach is geometric, based on submanifold geometry, and does not use any classification result. Our motivation for finding such a result came from homogenous spaces. In particular for explaining the inextendibility of the presentation group of an isotropy irreducible space which is not the sphere.

In the final part of this talk we will show these applications. First we will do some remarks for motivating why the only transitive group is the full orthogonal group.

This theorem was independently proved by Paul-A. Nagy by using the classification of the so-called Berger algebras, though in this final form a little earlier.

Our approach is geometric, based on submanifold geometry, and does not use any classification result. Our motivation for finding such a result came from homogenous spaces. In particular for explaining the inextendibility of the presentation group of an isotropy irreducible space which is not the sphere.

In the final part of this talk we will show these applications. First we will do some remarks for motivating why the only transitive group is the full orthogonal group.

From a skew-torsion holonomy system $[\mathbb{V}, \theta, G]$ one can construct a holonomy system $[\mathbb{V}, R, G]$, with $sc(R) \neq 0$

Namely, by defining

$$R_{x.y} = [\theta_x, \theta_y] - \frac{2}{3}(\theta_x.\theta)_y$$

Observe that

$[\mathbb{V}, \theta, G]$ symmetric

$$\Rightarrow$$

$\theta_x.\theta = 0$ derivation

From a skew-torsion holonomy system $[\mathbb{V}, \theta, G]$ one can construct a holonomy system $[\mathbb{V}, R, G]$, with $sc(R) \neq 0$

Namely, by defining

$$R_{x,y} = [\theta_x, \theta_y] - \frac{2}{3}(\theta_x.\theta)_y$$

Observe that

$$[\mathbb{V}, \theta, G] \ \text{ symmetric}$$

$$\Rightarrow$$

$$\theta_x.\theta = 0 \quad \text{derivation}$$

From a skew-torsion holonomy system $[\mathbb{V}, \theta, G]$ one can construct a holonomy system $[\mathbb{V}, R, G]$, with $sc(R) \neq 0$

Namely, by defining

$$R_{x.y} = [\theta_x, \theta_y] - \frac{2}{3}(\theta_x.\theta)_y$$

Observe that

$[\mathbb{V}, \theta, G]$ symmetric

$\Rightarrow$

$\theta_x.\theta = 0$ derivation

Introduction.

Riemannian holonomy.

Submanifold geometry and holonomy.

Holonomy systems and skew-torsion holonomy systems.

Applications to naturally reductive spaces.

From a skew-torsion holonomy system $[\mathbb{V}, \theta, G]$ one can construct a holonomy system $[\mathbb{V}, R, G]$, with $sc(R) \neq 0$

Namely, by defining

$$R_{x.y} = [\theta_x, \theta_y] - \frac{2}{3}(\theta_x.\theta)_y$$

Observe that

$$[\mathbb{V}, \theta, G] \ \text{ symmetric}$$

$$\Rightarrow$$

$$\theta_x.\theta = 0 \quad \text{derivation}$$

Introduction.

Riemannian holonomy.

Submanifold geometry and holonomy.

Holonomy systems and skew-torsion holonomy systems.

Applications to naturally reductive spaces.

From a skew-torsion holonomy system $[\mathbb{V}, \theta, G]$ one can construct a holonomy system $[\mathbb{V}, R, G]$, with $sc(R) \neq 0$

Namely, by defining

$$R_{x.y} = [\theta_x, \theta_y] - \frac{2}{3}(\theta_x.\theta)_y$$

Observe that

$$[\mathbb{V}, \theta, G] \ \text{ symmetric}$$

$$\Rightarrow$$

$$\theta_x.\theta = 0 \quad \text{derivation}$$

From a skew-torsion holonomy system $[\mathbb{V}, \theta, G]$ one can construct a holonomy system $[\mathbb{V}, R, G]$, with $sc(R) \neq 0$

Namely, by defining

$$R_{x.y} = [\theta_x, \theta_y] - \frac{2}{3}(\theta_x.\theta)_y$$

Observe that

$$[\mathbb{V}, \theta, G] \ \text{ symmetric}$$

$$\Rightarrow$$

$$\theta_x.\theta = 0 \quad \text{derivation}$$

From a skew-torsion holonomy system $[\mathbb{V}, \theta, G]$ one can construct a holonomy system $[\mathbb{V}, R, G]$, with $sc(R) \neq 0$

Namely, by defining

$$R_{x.y} = [\theta_x, \theta_y] - \frac{2}{3}(\theta_x.\theta)_y$$

Observe that

$$[\mathbb{V}, \theta, G] \ \text{ symmetric}$$

$$\Rightarrow$$

$$\theta_x.\theta = 0 \quad \text{derivation}$$

From a symmetric holonomy system one can construct a symmetric space. From a symmetric skew-torsion holonomy system one can build up a compact Lie group, by defining on $\mathcal{G}$ the bracket

$$[x, y] = \theta_x y$$

In some sense, skew-torsion holonomy systems are to holonomy systems what compact Lie groups are to compact symmetric spaces.

This suggest that a transitive skew-torsion holonomy system must have $G = SO(\mathbb{V})$, since the only compact rank one Lie group is, up to a cover, $SO(3)$ (whose isotropy representation is the standard representation of $SO(3)$).

From a symmetric holonomy system one can construct a symmetric space. From a symmetric skew-torsion holonomy system one can build up a compact Lie group, by defining on $\mathcal{G}$ the bracket

$$[x, y] = \theta_x y$$

In some sense, skew-torsion holonomy systems are to holonomy systems what compact Lie groups are to compact symmetric spaces.

This suggest that a transitive skew-torsion holonomy system must have $G = SO(\mathbb{V})$, since the only compact rank one Lie group is, up to a cover, $SO(3)$ (whose isotropy representation is the standard representation of $SO(3)$).

Introduction.

Riemannian holonomy.

Submanifold geometry and holonomy.

Holonomy systems and skew-torsion holonomy systems.

Applications to naturally reductive spaces.

From a symmetric holonomy system one can construct a symmetric space. From a symmetric skew-torsion holonomy system one can build up a compact Lie group, by defining on $\mathcal{G}$ the bracket

$$[x, y] = \theta_x y$$

In some sense, skew-torsion holonomy systems are to holonomy systems what compact Lie groups are to compact symmetric spaces.

This suggest that a transitive skew-torsion holonomy system must have $G = SO(\mathbb{V})$, since the only compact rank one Lie group is, up to a cover, $SO(3)$ (whose isotropy representation is the standard representation of $SO(3)$).

From a symmetric holonomy system one can construct a symmetric space. From a symmetric skew-torsion holonomy system one can build up a compact Lie group, by defining on $\mathcal{G}$ the bracket

$$[x, y] = \theta_x y$$

In some sense, skew-torsion holonomy systems are to holonomy systems what compact Lie groups are to compact symmetric spaces.

This suggest that a transitive skew-torsion holonomy system must have $G = SO(\mathbb{V})$, since the only compact rank one Lie group is, up to a cover, $SO(3)$ (whose isotropy representation is the standard representation of $SO(3)$).

From a symmetric holonomy system one can construct a symmetric space. From a symmetric skew-torsion holonomy system one can build up a compact Lie group, by defining on $\mathcal{G}$ the bracket

$$[x, y] = \theta_x y$$

In some sense, skew-torsion holonomy systems are to holonomy systems what compact Lie groups are to compact symmetric spaces.

This suggest that a transitive skew-torsion holonomy system must have $G = SO(\mathbb{V})$, since the only compact rank one Lie group is, up to a cover, $SO(3)$ (whose isotropy representation is the standard representation of $SO(3)$).

From a symmetric holonomy system one can construct a symmetric space. From a symmetric skew-torsion holonomy system one can build up a compact Lie group, by defining on $\mathcal{G}$ the bracket

$$[x, y] = \theta_x y$$

In some sense, skew-torsion holonomy systems are to holonomy systems what compact Lie groups are to compact symmetric spaces.

This suggest that a transitive skew-torsion holonomy system must have $G = SO(\mathbb{V})$, since the only compact rank one Lie group is, up to a cover, $SO(3)$ (whose isotropy representation is the standard representation of $SO(3)$).

Nautrally reductive spaces.

Let $M = G/H$ be a homogeneous compact Riemannian manifold with a G-invariant metric $\langle \ , \ \rangle$. The space M is said to be naturally reductive if there exists a reductive decomposition

$$\mathcal{G} = \mathfrak{h} \oplus \mathfrak{m}$$

$\mathcal{G} = \mathrm{Lie}(G)$, $\mathfrak{h} = \mathrm{Lie}(H)$ and $\mathrm{Ad}(H)(\mathfrak{m}) \subset \mathfrak{m}$.

such that the geodesics by $p = [e]$ are given by

$$\gamma_{X.p}(t) = \mathrm{Exp}(tX).p$$

In other words, the geodesics of M coincides with the ∇^c-geodesics (i.e., the geodesics with respect the canonical connection ∇^c, associated to the reductive decomposition).

Nautrally reductive spaces.

Let $M = G/H$ be a homogeneous compact Riemannian manifold with a G-invariant metric $\langle\ ,\ \rangle$. The space M is said to be naturally reductive if there exists a reductive decomposition

$$\mathcal{G} = \mathfrak{h} \oplus \mathfrak{m}$$

$\mathcal{G} = \mathrm{Lie}(G)$, $\mathfrak{h} = \mathrm{Lie}(H)$ and $\mathrm{Ad}(H)(\mathfrak{m}) \subset \mathfrak{m}$.

such that the geodesics by $p = [e]$ are given by

$$\gamma_{X.p}(t) = \mathrm{Exp}(tX).p$$

In other words, the geodesics of M coincides with the ∇^c-geodesics (i.e., the geodesics with respect the canonical connection ∇^c, associated to the reductive decomposition).

Nautrally reductive spaces.

Let $M = G/H$ be a homogeneous compact Riemannian manifold with a G-invariant metric $\langle \ , \ \rangle$. The space M is said to be naturally reductive if there exists a reductive decomposition

$$\mathcal{G} = \mathfrak{h} \oplus \mathfrak{m}$$

$\mathcal{G} = \mathrm{Lie}(G)$, $\mathfrak{h} = \mathrm{Lie}(H)$ and $\mathrm{Ad}(H)(\mathfrak{m}) \subset \mathfrak{m}$.

such that the geodesics by $p = [e]$ are given by

$$\gamma_{X.p}(t) = \mathrm{Exp}(tX).p$$

In other words, the geodesics of M coincides with the ∇^c-geodesics (i.e., the geodesics with respect the canonical connection ∇^c, associated to the reductive decomposition).

Nautrally reductive spaces.

Let $M = G/H$ be a homogeneous compact Riemannian manifold with a G-invariant metric $\langle\ ,\ \rangle$. The space M is said to be naturally reductive if there exists a reductive decomposition

$$\mathcal{G} = \mathfrak{h} \oplus \mathfrak{m}$$

$\mathcal{G} = \mathrm{Lie}(G)$, $\mathfrak{h} = \mathrm{Lie}(H)$ and $\mathrm{Ad}(H)(\mathfrak{m}) \subset \mathfrak{m}$.

such that the geodesics by $p = [e]$ are given by

$$\gamma_{X.p}(t) = \mathrm{Exp}(tX).p$$

In other words, the geodesics of M coincides with the ∇^c-geodesics (i.e., the geodesics with respect the canonical connection ∇^c, associated to the reductive decomposition).

Nautrally reductive spaces.

Let $M = G/H$ be a homogeneous compact Riemannian manifold with a G-invariant metric $\langle\ ,\ \rangle$. The space M is said to be naturally reductive if there exists a reductive decomposition

$$\mathcal{G} = \mathfrak{h} \oplus \mathfrak{m}$$

$\mathcal{G} = \mathrm{Lie}(G)$, $\mathfrak{h} = \mathrm{Lie}(H)$ and $\mathrm{Ad}(H)(\mathfrak{m}) \subset \mathfrak{m}$.

such that the geodesics by $p = [e]$ are given by

$$\gamma_{X.p}(t) = \mathrm{Exp}(tX).p$$

In other words, the geodesics of M coincides with the ∇^c-geodesics (i.e., the geodesics with respect the canonical connection ∇^c, associated to the reductive decomposition).

Introduction.

Riemannian holonomy.

Submanifold geometry and holonomy.

Holonomy systems and skew-torsion holonomy systems.

Applications to naturally reductive spaces.

Nautrally reductive spaces.

Let $M = G/H$ be a homogeneous compact Riemannian manifold with a G-invariant metric $\langle\ ,\ \rangle$. The space M is said to be naturally reductive if there exists a reductive decomposition

$$\mathcal{G} = \mathfrak{h} \oplus \mathfrak{m}$$

$\mathcal{G} = \mathrm{Lie}(G)$, $\mathfrak{h} = \mathrm{Lie}(H)$ and $\mathrm{Ad}(H)(\mathfrak{m}) \subset \mathfrak{m}$.

such that the geodesics by $p = [e]$ are given by

$$\gamma_{X.p}(t) = \mathrm{Exp}(tX).p$$

In other words, the geodesics of M coincides with the ∇^c-geodesics (i.e., the geodesics with respect the canonical connection ∇^c, associated to the reductive decomposition).

Nautrally reductive spaces.

Let $M = G/H$ be a homogeneous compact Riemannian manifold with a G-invariant metric $\langle\ ,\ \rangle$. The space M is said to be naturally reductive if there exists a reductive decomposition

$$\mathcal{G} = \mathfrak{h} \oplus \mathfrak{m}$$

$\mathcal{G} = \mathrm{Lie}(G)$, $\mathfrak{h} = \mathrm{Lie}(H)$ and $\mathrm{Ad}(H)(\mathfrak{m}) \subset \mathfrak{m}$.

such that the geodesics by $p = [e]$ are given by

$$\gamma_{X.p}(t) = \mathrm{Exp}(tX).p$$

In other words, the geodesics of M coincides with the ∇^c-geodesics (i.e., the geodesics with respect the canonical connection ∇^c, associated to the reductive decomposition).

Introduction.

Riemannian holonomy.

Submanifold geometry and holonomy.

Holonomy systems and skew-torsion holonomy systems.

Applications to naturally reductive spaces.

The Levi-Civita connection is given by

$$\nabla_v \tilde{w} = \frac{1}{2}[\tilde{v}, \tilde{w}]_p$$

and

$$\nabla^c_v \tilde{w} = [\tilde{v}, \tilde{w}]_p \quad ,$$

where, for $u \in T_pM$, $\tilde{u}$ is the Killing field on M induced by the unique $X \in \mathfrak{m}$ such that $X.p = u$.

So, the difference tensor between both connections, which is totally skew, is given by

$$D_v w = -\frac{1}{2}\nabla_v \tilde{w}$$

The Levi-Civita connection is given by

$$\nabla_v \tilde{w} = \frac{1}{2}[\tilde{v}, \tilde{w}]_p$$

and

$$\nabla^c_v \tilde{w} = [\tilde{v}, \tilde{w}]_p \quad ,$$

where, for $u \in T_pM$, $\tilde{u}$ is the Killing field on M induced by the unique $X \in \mathfrak{m}$ such that $X.p = u$.

So, the difference tensor between both connections, which is totally skew, is given by

$$D_v w = -\frac{1}{2}\nabla_v \tilde{w}$$

Introduction.

Riemannian holonomy.

Submanifold geometry and holonomy.

Holonomy systems and skew-torsion holonomy systems.

Applications to naturally reductive spaces.

The Levi-Civita connection is given by

$$\nabla_v \tilde{w} = \frac{1}{2}[\tilde{v}, \tilde{w}]_p$$

and

$$\nabla^c_v \tilde{w} = [\tilde{v}, \tilde{w}]_p \quad ,$$

where, for $u \in T_pM$, $\tilde{u}$ is the Killing field on M induced by the unique $X \in \mathfrak{m}$ such that $X.p = u$.

So, the difference tensor between both connections, which is totally skew, is given by

$$D_v w = -\frac{1}{2}\nabla_v \tilde{w}$$

The Levi-Civita connection is given by

$$\nabla_v \tilde{w} = \frac{1}{2}[\tilde{v}, \tilde{w}]_p$$

and

$$\nabla^c_v \tilde{w} = [\tilde{v}, \tilde{w}]_p \quad ,$$

where, for $u \in T_pM$, $\tilde{u}$ is the Killing field on M induced by the unique $X \in \mathfrak{m}$ such that $X.p = u$.

So, the difference tensor between both connections, which is totally skew, is given by

$$D_v w = -\frac{1}{2}\nabla_v \tilde{w}$$

Any isometry $g \in \mathrm{Iso}(M)_p$ (full isotropy at p) maps ∇ into itself, but, in general, maps ∇^c into another canonical connection ${}^g\nabla^c$, associated to the reductive decomposition

$$\mathcal{G}^g = \mathfrak{h}^g \oplus \mathfrak{m}^g$$

$\mathcal{G}^g = \mathrm{Ad}(g)\mathcal{G}$, $\mathfrak{h}^g = \mathrm{Ad}(g)\mathfrak{h}$, $\mathfrak{m}^g = \mathrm{Ad}(g)\mathfrak{m}$

Any isometry $g \in \mathrm{Iso}(M)_9$ (full isotropy at p) maps ∇ into itself, but, in general, maps ∇^c into another canonical connection ${}^g\nabla^c$, associated to the reductive decomposition

$$\mathcal{G}^g = \mathfrak{h}^g \oplus \mathfrak{m}^g$$

$\mathcal{G}^g = \mathrm{Ad}(g)\mathcal{G}$, $\mathfrak{h}^g = \mathrm{Ad}(g)\mathfrak{h}$, $\mathfrak{m}^g = \mathrm{Ad}(g)\mathfrak{m}$

Introduction.

Riemannian holonomy.

Submanifold geometry and holonomy.

Holonomy systems and skew-torsion holonomy systems.

Applications to naturally reductive spaces.

Any isometry $g \in \mathrm{Iso}(M)_9$ (full isotropy at p) maps ∇ into itself, but, in general, maps ∇^c into another canonical connection ${}^g\nabla^c$, associated to the reductive decomposition

$$\mathcal{G}^g = \mathfrak{h}^g \oplus \mathfrak{m}^g$$

$\mathcal{G}^g = \mathrm{Ad}(g)\mathcal{G}$, $\mathfrak{h}^g = \mathrm{Ad}(g)\mathfrak{h}$, $\mathfrak{m}^g = \mathrm{Ad}(g)\mathfrak{m}$

Any isometry $g \in \mathrm{Iso}(M)_{\mathfrak{p}}$ (full isotropy at p) maps ∇ into itself, but, in general, maps ∇^c into another canonical connection ${}^g\nabla^c$, associated to the reductive decomposition

$$\mathcal{G}^g = \mathfrak{h}^g \oplus \mathfrak{m}^g$$

$\mathcal{G}^g = \mathrm{Ad}(g)\mathcal{G}$, $\mathfrak{h}^g = \mathrm{Ad}(g)\mathfrak{h}$, $\mathfrak{m}^g = \mathrm{Ad}(g)\mathfrak{m}$

We have that

$$D^g_v w = -\frac{1}{2}\nabla_v \tilde{w}^g$$

where $D^g = \nabla - \nabla^g$ and $\tilde{w}^g$ is the Killing field on M induced by the unique $X \in \mathfrak{m}^g$ with $X.p = w$.
Then

$$\theta_v w := D_v w - D^g_v w = -\frac{1}{2}\nabla_v Z$$

where $Z = \tilde{w} - \tilde{w}^g$.

Observe that $Z_p = 0$. Then ∇Z lies in the (full) isotropy algebra. Thus θ takes values in the isotropy algebra.

Combining this last observation with the skew- torsion holonomy theorem and other geometric arguments one has the following.

We have that

$$D^g_v w = -\frac{1}{2}\nabla_v \tilde{w}^g$$

where $D^g = \nabla - \nabla^g$ and $\tilde{w}^g$ is the Killing field on M induced by the unique $X \in \mathfrak{m}^g$ with $X.p = w$.
Then

$$\theta_v w := D_v w - D^g_v w = -\frac{1}{2}\nabla_v Z$$

where $Z = \tilde{w} - \tilde{w}^g$.

Observe that $Z_p = 0$. Then ∇Z lies in the (full) isotropy algebra. Thus θ takes values in the isotropy algebra.

Combining this last observation with the skew- torsion holonomy theorem and other geometric arguments one has the following.

Introduction.

Riemannian holonomy.

Submanifold geometry and holonomy.

Holonomy systems and skew-torsion holonomy systems.

Applications to naturally reductive spaces.

We have that

$$D^{g}_{v} w = -\frac{1}{2} \nabla_{v} \tilde{w}^{g}$$

where $D^{g} = \nabla - \nabla^{g}$ and $\tilde{w}^{g}$ is the Killing field on M induced by the unique $X \in \mathfrak{m}^{g}$ with $X.p = w$.

Then

$$\theta_{v} w := D_{v} w - D^{g}_{v} w = -\frac{1}{2} \nabla_{v} Z$$

where $Z = \tilde{w} - \tilde{w}^{g}$.

Observe that $Z_{p} = 0$. Then ∇Z lies in the (full) isotropy algebra. Thus θ takes values in the isotropy algebra.

Combining this last observation with the skew- torsion holonomy theorem and other geometric arguments one has the following.

We have that

$$D^g_v w = -\frac{1}{2}\nabla_v \tilde{w}^g$$

where $D^g = \nabla - \nabla^g$ and $\tilde{w}^g$ is the Killing field on M induced by the unique $X \in \mathfrak{m}^g$ with $X.p = w$.
Then

$$\theta_v w := D_v w - D^g_v w = -\frac{1}{2}\nabla_v Z$$

where $Z = \tilde{w} - \tilde{w}^g$.

Observe that $Z_p = 0$. Then ∇Z lies in the (full) isotropy algebra. Thus θ takes values in the isotropy algebra.

Combining this last observation with the skew- torsion holonomy theorem and other geometric arguments one has the following.

We have that

$$D^g_v w = -\frac{1}{2}\nabla_v \tilde{w}^g$$

where $D^g = \nabla - \nabla^g$ and $\tilde{w}^g$ is the Killing field on M induced by the unique $X \in \mathfrak{m}^g$ with $X.p = w$.
Then

$$\theta_v w := D_v w - D^g_v w = -\frac{1}{2}\nabla_v Z$$

where $Z = \tilde{w} - \tilde{w}^g$.

Observe that $Z_p = 0$. Then ∇Z lies in the (full) isotropy algebra. Thus θ takes values in the isotropy algebra.

Combining this last observation with the skew- torsion holonomy theorem and other geometric arguments one has the following.

Introduction.

Riemannian holonomy.

Submanifold geometry and holonomy.

Holonomy systems and skew-torsion holonomy systems.

Applications to naturally reductive spaces.

We have that

$$D^g_v w = -\frac{1}{2}\nabla_v \tilde{w}^g$$

where $D^g = \nabla - \nabla^g$ and $\tilde{w}^g$ is the Killing field on M induced by the unique $X \in \mathfrak{m}^g$ with $X.p = w$.
Then

$$\theta_v w := D_v w - D^g_v w = -\frac{1}{2}\nabla_v Z$$

where $Z = \tilde{w} - \tilde{w}^g$.

Observe that $Z_p = 0$. Then ∇Z lies in the (full) isotropy algebra. Thus θ takes values in the isotropy algebra.

Combining this last observation with the skew- torsion holonomy theorem and other geometric arguments one has the following.

Introduction.

Riemannian holonomy.

Submanifold geometry and holonomy.

Holonomy systems and skew-torsion holonomy systems.

Applications to naturally reductive spaces.

We have that

$$D^g_v w = -\frac{1}{2}\nabla_v \tilde{w}^g$$

where $D^g = \nabla - \nabla^g$ and $\tilde{w}^g$ is the Killing field on M induced by the unique $X \in \mathfrak{m}^g$ with $X.p = w$.
Then

$$\theta_v w := D_v w - D^g_v w = -\frac{1}{2}\nabla_v Z$$

where $Z = \tilde{w} - \tilde{w}^g$.

Observe that $Z_p = 0$. Then ∇Z lies in the (full) isotropy algebra. Thus θ takes values in the isotropy algebra.

Combining this last observation with the skew- torsion holonomy theorem and other geometric arguments one has the following.

Theorem (O.-Reggiani). *Let M be a compact naturally reductive Riemannian manifold, which is locally irreducible. Assume furthermore that M is neither (globally) isometric to a sphere, nor to a real projective space, nor to a compact simple Lie group with a bi-invariant metric. Then the canonical connection is unique.*

Corollary (O.-Reggiani). *Let $M = G/H$ be a naturally reductive Riemannian manifold and let ∇^c be the associated canonical connection. Assume that M is locally irreducible and that $M \neq S^n$, $M \neq \mathbb{R}P^n$. Then*

1. $\mathrm{Iso}_0(M) = \mathrm{Aff}_0(M, \nabla^c)$.
2. *If* $\mathrm{Iso}(M) \not\subset \mathrm{Aff}(M, \nabla^c)$ *then M is isometric to a simple Lie group, endowed with a bi-invariant metric (and in this case, the geodesic symmetry maps ∇^c into $2\nabla - \nabla^c$).*

Theorem (O.-Reggiani). *Let M be a compact naturally reductive Riemannian manifold, which is locally irreducible. Assume furthermore that M is neither (globally) isometric to a sphere, nor to a real projective space, nor to a compact simple Lie group with a bi-invariant metric. Then the canonical connection is unique.*

Corollary (O.-Reggiani). *Let $M = G/H$ be a naturally reductive Riemannian manifold and let ∇^c be the associated canonical connection. Assume that M is locally irreducible and that $M \neq S^n$, $M \neq \mathbb{R}P^n$. Then*

1. $\mathrm{Iso}_0(M) = \mathrm{Aff}_0(M, \nabla^c)$.
2. *If* $\mathrm{Iso}(M) \not\subset \mathrm{Aff}(M, \nabla^c)$ *then M is isometric to a simple Lie group, endowed with a bi-invariant metric (and in this case, the geodesic symmetry maps ∇^c into $2\nabla - \nabla^c$).*

Theorem (O.-Reggiani). *Let M be a compact naturally reductive Riemannian manifold, which is locally irreducible. Assume furthermore that M is neither (globally) isometric to a sphere, nor to a real projective space, nor to a compact simple Lie group with a bi-invariant metric. Then the canonical connection is unique.*

Corollary (O.-Reggiani). *Let $M = G/H$ be a naturally reductive Riemannian manifold and let ∇^c be the associated canonical connection. Assume that M is locally irreducible and that $M \neq S^n$, $M \neq \mathbb{R}P^n$. Then*

1. $\mathrm{Iso}_0(M) = \mathrm{Aff}_0(M, \nabla^c)$.
2. *If $\mathrm{Iso}(M) \not\subset \mathrm{Aff}(M, \nabla^c)$ then M is isometric to a simple Lie group, endowed with a bi-invariant metric (and in this case, the geodesic symmetry maps ∇^c into $2\nabla - \nabla^c$).*

Theorem (O.-Reggiani). *Let M be a compact naturally reductive Riemannian manifold, which is locally irreducible. Assume furthermore that M is neither (globally) isometric to a sphere, nor to a real projective space, nor to a compact simple Lie group with a bi-invariant metric. Then the canonical connection is unique.*

Corollary (O.-Reggiani). *Let $M = G/H$ be a naturally reductive Riemannian manifold and let ∇^c be the associated canonical connection. Assume that M is locally irreducible and that $M \neq S^n$, $M \neq \mathbb{R}P^n$. Then*

1. $\mathrm{Iso}_0(M) = \mathrm{Aff}_0(M, \nabla^c)$.
2. *If* $\mathrm{Iso}(M) \not\subset \mathrm{Aff}(M, \nabla^c)$ *then M is isometric to a simple Lie group, endowed with a bi-invariant metric (and in this case, the geodesic symmetry maps ∇^c into $2\nabla - \nabla^c$).*

The above theorem has also the following corollary that explains, in a geometric way, that the presentation of an isotropy irreducible space gives the connected component of the full isometry group. This question was posed by J. Wolf, for the strongly isotropy irreducible spaces, and by M. Wang and W. Ziller in general.

Corollary (Wolf; Wang-Ziller). *Let $M^n = G/H$ be a compact, simply connected, irreducible homogeneous Riemannian manifold such that M is not isometric to the sphere S^n. Assume that M is isotropy irreducible with respect to the pair (G, H) (effective action). Assume, furthermore, that M is not isometric to a (simple) compact Lie group with a bi-invariant metric. Then $G_0 = \mathrm{Iso}_0(M)$.*

For a compact normal homogeneous space (i.e. $\mathfrak{m} = \mathfrak{h}^\perp$), with G simple, this result was obtained by classification by Onishchik. For homogeneous manifolds of positive curvature the full isometry group was determined by Shankar.

The above theorem has also the following corollary that explains, in a geometric way, that the presentation of an isotropy irreducible space gives the connected component of the full isometry group. This question was posed by J. Wolf, for the strongly isotropy irreducible spaces, and by M. Wang and W. Ziller in general.

Corollary (Wolf; Wang-Ziller). *Let $M^n = G/H$ be a compact, simply connected, irreducible homogeneous Riemannian manifold such that M is not isometric to the sphere S^n. Assume that M is isotropy irreducible with respect to the pair (G, H) (effective action). Assume, furthermore, that M is not isometric to a (simple) compact Lie group with a bi-invariant metric. Then $G_0 = \mathrm{Iso}_0(M)$.*

For a compact normal homogeneous space (i.e. $\mathfrak{m} = \mathfrak{h}^\perp$), with G simple, this result was obtained by classification by Onishchik. For homogeneous manifolds of positive curvature the full isometry group was determined by Shankar.

The above theorem has also the following corollary that explains, in a geometric way, that the presentation of an isotropy irreducible space gives the connected component of the full isometry group. This question was posed by J. Wolf, for the strongly isotropy irreducible spaces, and by M. Wang and W. Ziller in general.

Corollary (Wolf; Wang-Ziller). *Let $M^n = G/H$ be a compact, simply connected, irreducible homogeneous Riemannian manifold such that M is not isometric to the sphere S^n. Assume that M is isotropy irreducible with respect to the pair (G, H) (effective action). Assume, furthermore, that M is not isometric to a (simple) compact Lie group with a bi-invariant metric. Then $G_0 = \mathrm{Iso}_0(M)$.*

For a compact normal homogeneous space (i.e. $\mathfrak{m} = \mathfrak{h}^\perp$), with G simple, this result was obtained by classification by Onishchik. For homogeneous manifolds of positive curvature the full isometry group was determined by Shankar.

Introduction.

Riemannian holonomy.

Submanifold geometry and holonomy.

Holonomy systems and skew-torsion holonomy systems.

Applications to naturally reductive spaces.

The above theorem has also the following corollary that explains, in a geometric way, that the presentation of an isotropy irreducible space gives the connected component of the full isometry group. This question was posed by J. Wolf, for the strongly isotropy irreducible spaces, and by M. Wang and W. Ziller in general.

Corollary (Wolf; Wang-Ziller). *Let $M^n = G/H$ be a compact, simply connected, irreducible homogeneous Riemannian manifold such that M is not isometric to the sphere S^n. Assume that M is isotropy irreducible with respect to the pair (G, H) (effective action). Assume, furthermore, that M is not isometric to a (simple) compact Lie group with a bi-invariant metric. Then $G_0 = \mathrm{Iso}_0(M)$.*

For a compact normal homogeneous space (i.e. $\mathfrak{m} = \mathfrak{h}^\perp$), with G simple, this result was obtained by classification by Onishchik. For homogeneous manifolds of positive curvature the full isometry group was determined by Shankar.

The above theorem has also the following corollary that explains, in a geometric way, that the presentation of an isotropy irreducible space gives the connected component of the full isometry group. This question was posed by J. Wolf, for the strongly isotropy irreducible spaces, and by M. Wang and W. Ziller in general.

Corollary (Wolf; Wang-Ziller). *Let $M^n = G/H$ be a compact, simply connected, irreducible homogeneous Riemannian manifold such that M is not isometric to the sphere S^n. Assume that M is isotropy irreducible with respect to the pair (G, H) (effective action). Assume, furthermore, that M is not isometric to a (simple) compact Lie group with a bi-invariant metric. Then $G_0 = \mathrm{Iso}_0(M)$.*

For a compact normal homogeneous space (i.e. $\mathfrak{m} = \mathfrak{h}^\perp$), with G simple, this result was obtained by classification by Onishchik. For homogeneous manifolds of positive curvature the full isometry group was determined by Shankar.

The above theorem has also the following corollary that explains, in a geometric way, that the presentation of an isotropy irreducible space gives the connected component of the full isometry group. This question was posed by J. Wolf, for the strongly isotropy irreducible spaces, and by M. Wang and W. Ziller in general.

Corollary (Wolf; Wang-Ziller). *Let $M^n = G/H$ be a compact, simply connected, irreducible homogeneous Riemannian manifold such that M is not isometric to the sphere S^n. Assume that M is isotropy irreducible with respect to the pair (G, H) (effective action). Assume, furthermore, that M is not isometric to a (simple) compact Lie group with a bi-invariant metric. Then $G_0 = \mathrm{Iso}_0(M)$.*

For a compact normal homogeneous space (i.e. $\mathfrak{m} = \mathfrak{h}^\perp$), with G simple, this result was obtained by classification by Onishchik. For homogeneous manifolds of positive curvature the full isometry group was determined by Shankar.

The above theorem has also the following corollary that explains, in a geometric way, that the presentation of an isotropy irreducible space gives the connected component of the full isometry group. This question was posed by J. Wolf, for the strongly isotropy irreducible spaces, and by M. Wang and W. Ziller in general.

Corollary (Wolf; Wang-Ziller). *Let $M^n = G/H$ be a compact, simply connected, irreducible homogeneous Riemannian manifold such that M is not isometric to the sphere S^n. Assume that M is isotropy irreducible with respect to the pair (G, H) (effective action). Assume, furthermore, that M is not isometric to a (simple) compact Lie group with a bi-invariant metric. Then $G_0 = \mathrm{Iso}_0(M)$.*

For a compact normal homogeneous space (i.e. $\mathfrak{m} = \mathfrak{h}^\perp$), with G simple, this result was obtained by classification by Onishchik. For homogeneous manifolds of positive curvature the full isometry group was determined by Shankar.

The above theorem has also the following corollary that explains, in a geometric way, that the presentation of an isotropy irreducible space gives the connected component of the full isometry group. This question was posed by J. Wolf, for the strongly isotropy irreducible spaces, and by M. Wang and W. Ziller in general.

Corollary (Wolf; Wang-Ziller). *Let $M^n = G/H$ be a compact, simply connected, irreducible homogeneous Riemannian manifold such that M is not isometric to the sphere S^n. Assume that M is isotropy irreducible with respect to the pair (G, H) (effective action). Assume, furthermore, that M is not isometric to a (simple) compact Lie group with a bi-invariant metric. Then $G_0 = \mathrm{Iso}_0(M)$.*

For a compact normal homogeneous space (i.e. $\mathfrak{m} = \mathfrak{h}^{\perp}$), with G simple, this result was obtained by classification by Onishchik. For homogeneous manifolds of positive curvature the full isometry group was determined by Shankar.

By using the skew-torsion holonomy theorem, Silvio Reggiani computed explicitly (the connected component of) the full isometry group of a normal homogeneous space. Namely, if $M = G/H$ is such a space and S is the (connected component of the) fixed set of the isotropy H, regarded as a Lie group, then

$$\mathrm{Iso}(M)_0 = G_1 \times S$$

where G_1 is the semisimple part of G.

For the general naturally reductive case one has to replace S by an appropriate subgroup.

By using the skew-torsion holonomy theorem, Silvio Reggiani computed explicitly (the connected component of) the full isometry group of a normal homogeneous space. Namely, if $M = G/H$ is such a space and S is the (connected component of the) fixed set of the isotropy H, regarded as a Lie group, then

$$\mathrm{Iso}(M)_0 = G_1 \times S$$

where G_1 is the semisimple part of G.

For the general naturally reductive case one has to replace S by an appropriate subgroup.

By using the skew-torsion holonomy theorem, Silvio Reggiani computed explicitly (the connected component of) the full isometry group of a normal homogeneous space. Namely, if $M = G/H$ is such a space and S is the (connected component of the) fixed set of the isotropy H, regarded as a Lie group, then

$$\mathrm{Iso}(M)_0 = G_1 \times S$$

where G_1 is the semisimple part of G.

For the general naturally reductive case one has to replace S by an appropriate subgroup.

By using the skew-torsion holonomy theorem, Silvio Reggiani computed explicitly (the connected component of) the full isometry group of a normal homogeneous space. Namely, if $M = G/H$ is such a space and S is the (connected component of the) fixed set of the isotropy H, regarded as a Lie group, then

$$\mathrm{Iso}(M)_0 = G_1 \times S$$

where G_1 is the semisimple part of G.

For the general naturally reductive case one has to replace S by an appropriate subgroup.

Introduction.

Riemannian holonomy.

Submanifold geometry and holonomy.

Holonomy systems and skew-torsion holonomy systems.

Applications to naturally reductive spaces.

By using the skew-torsion holonomy theorem, Silvio Reggiani computed explicitly (the connected component of) the full isometry group of a normal homogeneous space. Namely, if $M = G/H$ is such a space and S is the (connected component of the) fixed set of the isotropy H, regarded as a Lie group, then

$$\mathrm{Iso}(M)_0 = G_1 \times S$$

where G_1 is the semisimple part of G.

For the general naturally reductive case one has to replace S by an appropriate subgroup.

By using the skew-torsion holonomy theorem, Silvio Reggiani computed explicitly (the connected component of) the full isometry group of a normal homogeneous space. Namely, if $M = G/H$ is such a space and S is the (connected component of the) fixed set of the isotropy H, regarded as a Lie group, then

$$\mathrm{Iso}(M)_0 = G_1 \times S$$

where G_1 is the semisimple part of G.

For the general naturally reductive case one has to replace S by an appropriate subgroup.

One also has the following

Theorem (O.- Reggiani). The holonomy group of an irreducible naturally reductive space, which is not symmetric, is the full (special) orthogonal group.

Question: to give an infinite dimensional version of the Berger Holonomy Theorem.

Many thanks!