FEATURE

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Research Area : Mathematics

From Algebras to Varieties

Derived categories after Grothendieck

Homological algebra is considered by mathematicians to be one of the most formal subjects within mathematics. Its formal austerity requires a lot of efforts to come through basic definitions and frightens off many of those who starts studying the subject. This high level of formality of the theory gives an impression that it is a *ding an sich*, something that has no possible way of comprehension for outsiders and efforts for learning the theory would never be paid back.

After Alexander Grothendieck, the great creator of modern homological algebra who introduced the concept of derived categories, had left the stage where he occupied a central place for decades, the opinion that the homological theory had reached its bounds and become a useless formal theory was widely spread in the mathematical community. Paradoxically, these sentiments were particularly strong in those countries, like France, where the influence of Grothendieck's ideas was particularly strong. When I visited Universities Paris 6/7 in the beginning of the new century, I was surprised to observe that Algebraic Geometry, the branch of mathematics where homological methods showed their extreme usefulness, was split in France into two separate trends: there were those who studied geometry by classical methods and did not want to hear anything about derived categories, and those who studied very formal aspects of derived categories and did not know anything about classical geometry of varieties superlatively developed by Italian school of the late 19th and early 20th century. Both groups had very strong representatives, but they had scarce overlap in research.

Perhaps, one of the reasons for this strange state of affairs was a side effect of the great achievement of one of the best Grothendieck's students, Pierre Deligne, who used complicated homological algebra to prove Weil conjectures. These conjectures are, probably, more of arithmetic nature and do not have so much to do with geometry of algebraic varieties in the classical sense of Italian school. For years, applications of derived categories were developed rather in the area of number theoretical and topological aspects of varieties and, later, in Representation Theory than in classical geometry of varieties.

Associative algebras

Let us trace back the idea of homological algebra on the example of categories of modules over an associative algebra. When mathematicians started to study various mathematical structures which admit operations like addition and product, they extracted an important class of such objects called associative rings. Basically, one requires that the addition is associative and commutative, the multiplication is associative too and multiplication is distributive with respect to addition:

$$(a+b)+c = a+(b+c),$$
$$a+b = b+a,$$

$$(ab)c = a(bc),$$
$$(a+b)c = ac+bc$$

If one adds the property that every element can be multiplied by scalars, then an associative ring becomes an associative algebra.

One can ask: why these conditions on operations have so omnipresent behavior in mathematics? Indeed, even if you start with another algebraic structure, like, for instance, Lie algebra, much of its theory, in particular, all relevant homological algebra, can be interpreted in terms of its universal enveloping algebra, which is an associative algebra. Well, the answer to this question is probably rather hard to formulate, though the question itself is very important.

When the theory, especially its homological aspects are developed far enough, one gets a need to extend the definition of associative algebra, and to generalize it to DG-algebras, A-infinity algebras, etc. When playing the game with various definitions, to have a clear reason for basic definitions is really crucial. The current state of affairs in homological algebra demands a clear understanding of the foundations of the theory. Associativity of multiplication is, basically, related to the fact that the composition of maps is an associative operation. Addition and multiplication by scalars come from the idea of *Linearization*, which seems to be based on the formalization of the "observation" that the space surrounding us (space-time) looks locally like a vector space. One can speculate that this subject is directly related to the basic principle of superposition in quantum mechanics, where linearity is the milestone. The need of DG-algebras and A-infinity algebras is a strong indication of homotopy flavor of the basic constructions in algebra. Recent development of the theory of types suggests that logical foundations of mathematics might also be naturally rooted in Homotopy Theory.

Homological Algebra

Let us see how mathematicians came to the idea of Homological Algebra. First, they observed that associative algebras are complicated objects. To understand why, one has to consider them in their "society," the place where they work, play tennis and communicate. This is the category of algebras, which means that associative algebras are observed together with ("communicate" by means of) maps between them, morphisms of algebras, i.e., maps

$$f: A \to B,$$

that preserve addition and multiplication. It would be easier to work with algebras if any such map has the kernel (elements that go to zero), the image (elements in B which come from A), and the quotient by the image to be of the same kind as algebras themselves. Both kernel and the image are subalgebras in A, but not every subalgebra can be the kernel of a morphism, it must be a so-called *ideal*, i.e., a subset which preserves multiplication by every element of the algebra. The image of the map is not usually an ideal, which prevents from forming the quotient of B by the image. All this makes studying of algebras a complicated story.

Now remember that associativity has come from composition of maps. This supports the idea to represent elements of algebra by maps in some space M. It is natural to assume that this space also has some linearity properties, like addition and multiplication by scalars. Thus, we come to the notion of module, or representation, over a given algebra.

"The society" of modules over a fixed algebra, i.e., the category of modules, is much better settled than that of algebras. The kernel of any morphism of modules over an algebra, as well as the quotient by the image is again a module over the same algebra. In other words, the category of modules over a fixed algebra is *Abelian*.

By reversing the ideology, it is reasonable to think that the algebras themselves are important only in what concerns their categories of modules. This leads to the notion of *Morita equivalence*. Two algebras are Morita equivalent if their categories of modules are equivalent. It is reasonable to adopt the viewpoint that algebras are important only because of their categories of modules. The fundamental idea of Category Theory is that we have to consider various structures like, for example, algebras or modules over algebras not as individual objects but rather as members of "societies," i.e., appropriate categories.

Now we are well-prepared to come to the basics of Homological Algebra. Consider a submodule K in module L. We know that there is a module M which is the quotient of L by K. We can think of L as being kind of split or decomposed into two simple pieces, K and M.

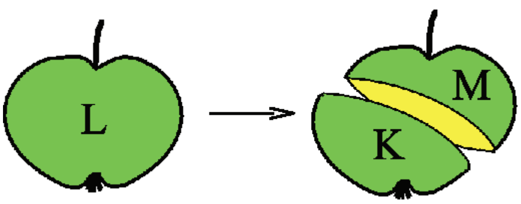

Figure 1. Splitting a module into a submodule and a factor-module.

Since the roles of K and M are clearly different, this situation is described by words "L is an extension of M by means of K." How can one describe all such extensions with fixed M and K?

The basic observation of homological algebra is that all such extensions are numbered by elements of an additive group which, moreover, admits multiplication by scalars! It was a surprising discovery that one can sum up two extensions with given M and K and obtain another extension with the same property. It is like conjurer's trick: he takes two apples, yellow and green, split each of them into two pieces, small and big, quickly mixes up all the pieces together by his hands, gets two new pieces, small and big, of a red apple, and finally joins these two pieces and shows a new one whole red apple.

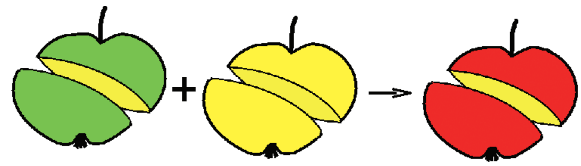

Figure 2. Addition of extensions.

The fascinating idea of derived categories is that one should enlarge the category of modules, the society where modules live, to include there "descendants and predecessors" so that extensions for given M and K could be interpreted as morphisms from M into the first descendant of K:

$$M \to K[1],$$

where K[1] is the notation for the first K-descendant (the "son" of K). The strange operation of summation of extensions then has the meaning of addition of morphisms which always exists in additive categories. This category which includes descendants of the original category was conceived by Alexander Grothendieck, who baptized it as *derived category*. The precise definition of the derived category uses old ideas of syzygies, or resolutions, in modern terminology, which go back to at least 19th century British mathematician Arthur Cayley and great German David Hilbert.

The idea of derived categories is, in fact, quite universal and applicable to many other mathematical theories, always, when objects of study comprise an Abelian category. The typical psychological problem for researchers is that when they study some particular area, for instance, complex analysis, with many technical details in its own and come to the point when they need to use homological methods, the idea of derived categories looks so perpendicular to their mathematical experience and so abstract and technical by itself, that a real courage is needed to plunge into this "hostile" sea with a hope to reach an island of interesting applications. I have only one advice for young researchers: start practicing to swim in this sea near the land when you are in kindergarten!

Derived categories of coherent sheaves

In algebraic geometry, objects like algebraic varieties, in their "social behavior," are similar to algebras, they don't comprise an Abelian category. So if we want to use Homological Algebra, we need to find an analogue of the category of modules. This is the category of coherent sheaves over an algebraic variety.

Let us look on what these sheaves are and how they naturally appear from the idea of *Linearization*. If we consider a smooth variety X, which is a very curvy-linear object, and look at it in a vicinity of some smooth subvariety Y, we will see that the variety is well-approximated by its linearized version, the normal bundle. Assume now that the variety is not smooth while the subvariety is, then the approximation would give us something which looks like "a vector bundle with a jump of dimension of some fibers." This is formalized as a coherent sheaf. As an example, consider a quadratic cone X and a line Y on X through the central point x of the cone. The fiber of the normal bundle is a line at every point of Y except the central point where the fiber is a plane, the vector space of dimension 2.

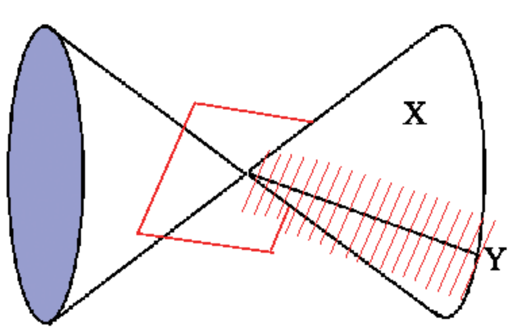

Figure 3. Jump of dimension of a fiber of the normal bundle to Y in X.

This jump of dimension of fibers of coherent sheaves is always located on a subvariety, say Y_1, of Y. Further higher rank jump of fibers might happen on a subvariety in Y_1, and so on.

It might be surprising that coherent sheaves are in a sense easier to tackle with than vector bundles. They are relatively tame features, because they live in the well-organized society, an Abelian category. So the machinery of derived categories is applicable to them: by including the "descendants" and "predecessors" we obtain the derived category of coherent sheaves.

Ideas of Grothendieck on derived categories were put on firm basis and vastly developed by many of his students, and first of all by Jean-Louis Verdier. Important formal properties of bounded derived categories of coherent sheaves on varieties, those derived categories which are most relevant to classical algebraic geometry, were scrutinized particularly in papers by Luc Illusie, but the structure of coherent derived categories of algebraic varieties remained totally obscure and their relation to geometry of varieties was unknown for decades.

Grothendieck's ideas were smuggled over the Iron Curtain into Russia by Yuri Manin during Russian political Thawing in Khrushchev era. Manin met with Grothendieck in the 60's and fully realized importance of these new homological ideas. Manin and his students and collaborators in Moscow explored the idea of derived categories and started to study derived categories of coherent sheaves for some algebraic varieties.

The situation in Moscow was similar to that in Paris: Manin's seminar studied formal, topological and arithmetic properties of algebraic varieties via derived categories, while Shafarevich's seminar and many representatives of his school, like Andrej Tyurin, Vassily Iskovskih and many others, worked on classical Algebraic Geometry in Italian style. They existed in parallel, though the splitting was not so profound as in France: it is sufficient to recall the outstanding achievement of Atiyah-Drinfeld-Hitchin-

Manin paper on classical geometry of instantons, which has clear homological flavor. It is also worth to mention that derived categories started to be actively applied to Representation Theory. One of the most spectacular achievements was the proof in the beginning of the 80's of Kazhdan-Lustig conjecture by Beilinson and Bernstein, also independently obtained by Brylinski and Kashiwara. But the structure of the derived categories of coherent sheaves was beyond the main stream.

There was a clear conceptual logic in studying derived categories and there was an evident deepness of results in the study of birational geometry and low dimensional varieties, though these results looked very miscellaneous and hard to grasp especially for the beginner, who I was by that time.

At some point, I realized that various contemporary developments in classical Algebraic Geometry might be approached via the derived category of coherent sheaves on an algebraic variety, if we consider the category as an invariant of the variety. Natural questions had quickly come. How to extract any information from this invariant? Is it possible to reconstruct usual invariants of varieties and vector bundles on them, like Hodge cohomology and Chern classes? Is it possible to reconstruct the variety itself from its derived category? How the derived category transforms under various geometric operations, for example birational transformations?

It appeared that the categories of some varieties had some bases, called exceptional collections, which are like orthonormal bases in a vector space with a scalar form. Though, the scalar form is rather non-symmetric and not skew symmetric, and semi-orthonormality is a more relevant analogy. A semi-orthonormal basis has an order on its elements. If you change the order and use the (semi-) orthonormalization process similar to Gramm-Schmidt orthonormalization, you will quickly come to the action of the braid group on the set of bases. This reflects a deep connection of derived categories to Homotopy Theory.

The problem of extracting any information from the derived category, when you consider it as an abstract triangulated category, comes from the fact that morphisms in this category don't have kernels and cokernels as they used to have in Abelian categories. A useful tool was discovered jointly with Mikhail Kapranov. It was Serre functor, the categorical incarnation of the canonical class of an algebraic variety. Using it, I was able to reconstruct the columns of Hodge diamond from the derived category and Chern character. It was really striking to see that derived invariants were columns and not rows of Hodge diamond, as it was standard "knowledge" that the rows made good sense as they were responsible for singular homology of the variety. The mirror symmetry conjecture appeared by that time with rows and columns of 3-dimensional Calabi-Yau varieties exchanged under the symmetry. I have conjectured that derived categories should play the central role in mirror symmetry. This was later formulated in a more precise form in Homological Mirror Symmetry by Maxim Kontsevich who proposed to compare the derived category of a complex variety with Fukaya category of the symplectic manifold on the other side of the mirror.

Jointly with Dmitry Orlov, we proved that the variety can be reconstructed from the derived category under the condition that the variety has ample canonical or anti-canonical class. On the other hand, examples of derived equivalences had already been known from papers of Shigeru Mukai for K3 surfaces and Abelian varieties. We found that derived categories have nice behavior under some special birational transformation in the Minimal Model Program of Shigefumi Mori. We conjectured that they are equivalent under so-called flop transformations, while flips should induce fully faithful functors between derived categories. This gave a new perspective to the program by interpreting that it is the derived category that should be nicely minimized in an appropriate sense. There are results by Tom Bridgeland, Yujiro Kawamata and others in favor of the conjecture, but the proof is far from being achieved yet.

When we consider the derived category as the primary invariant of an algebraic variety, we naturally come to the question what are the properties of categories which distinguish the class of derived categories of coherent sheaves on smooth algebraic varieties. Some nice properties of these categories were found relatively quickly in collaboration with Mikhail Kapranov and, later, with Michel Van den Bergh. These properties are also enjoyed by algebraic spaces, a modest extension of the class of algebraic varieties. Bertrand Toen and Michel Vaquie proved a nice theorem that if the derived category of any complex manifold satisfies those properties then the manifold must be an algebraic space.

On the other hand, it was fairly clear from the very beginning that there was no easy formulated property of the abstract category which would distinguish the class of derived categories of coherent sheaves. The idea came to me in the early 90's that we should regard all categories which satisfied good properties mentioned above, but which are not derived categories of geometric objects, as categorical images of non-commutative varieties. Despite of a number of results in this direction, for instance, a classification of noncommutative projective planes in a joint work with Alexander Polishchuk (inspired by an early work of Artin, Tate and Van den Bergh), and works on noncommutative blow-ups by Van den Bergh, Stafford and others, we are still very far from comprehending geometry of these noncommutative categorical varieties as compared to results in the commutative case. Better understanding of invariants of categories, similar to Hodge cohomology for commutative varieties, would certainly help to this end.

Looking forward, it seems reasonable to consider the category of all smooth algebraic varieties and fully faithful functors as morphisms between them, with possibly reasonable mild extensions for both objects and morphisms, and try to grasp the structure of this "society" by means of Homotopy Theory. Objects of the derived categories of coherent sheaves have interpretation as boundary conditions for B-models in Topological String Theory. Thus, appropriate understanding of this structure would give an insight in the landscape of possible compactifications of the stringy space-time.