FEATURE

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Research Area: Mathematics

Intersection Numbers and Differential Equations

Introduction

One of the most fundamental concepts in modern geometry is the notion of a manifold. It is very unlikely that someone who did not get a special training in mathematics has ever heard this word before. I am going to explain what a manifold is, but to begin with we can think of it as a specific part of the space around us, such as the circle that a little girl drew on the wall while her parents were not watching, the surface of a soccer ball, and even the entire Universe. Sometimes the shape of the manifold is easy to imagine, because we can see it, but most of the times it is impossible. For example, we can see only a small piece of the Universe. It looks like a box, but the true shape might be quite different. One mathematical approach to deal with things that we can't imagine is to find numbers, usually called invariants that capture as many geometrical properties as possible. The invariants that were studied quite extensively in the last 20 years are the so-called Gromov-Witten invariants. Although their origin goes back to classical problems in enumerative algebraic geometry, it is the recent developments in string theory that made them very interesting. The goal of string theory is to unify quantum mechanics and gravity.
Its main idea is to model particles by little strings. In particular, trajectories are not lines but surfaces. That is why the problem of determining what types and how many surfaces exist in a given manifold becomes very important in physics as well. I would like to write about one of the striking predictions of string theory, which has a unifying power in a sense that it suggests a relation between two quite different areas in Mathematics.

What is a manifold?

The basic examples are called linear vector spaces. The examples that we can imagine are the line, the plane, and the 3-dimensional space. Alternatively, we can think of these spaces in terms of coordinates. Namely, we draw a coordinate system by choosing an arbitrary point as an origin and 1, 2, or 3 orthogonal axes. Every point has coordinates that correspond to projecting the point to each coordinate axes. This way the line is the same as all real numbers. The plane is the same as all pairs of real numbers, while the 3-dimensional space is all triples of numbers. The dimension corresponds to the number of coordinates. Our imagination cannot go beyond dimension 3, so that we have no way to say what a 4-dimensional linear vector space is except that it is just all quadruples of real numbers.

The manifold is made from linear vector spaces by gluing. The simplest example is the circle, but let me explain the next simplest example the sphere, because it is more relevant to us. If we remove the North Pole N of the sphere then every other point P

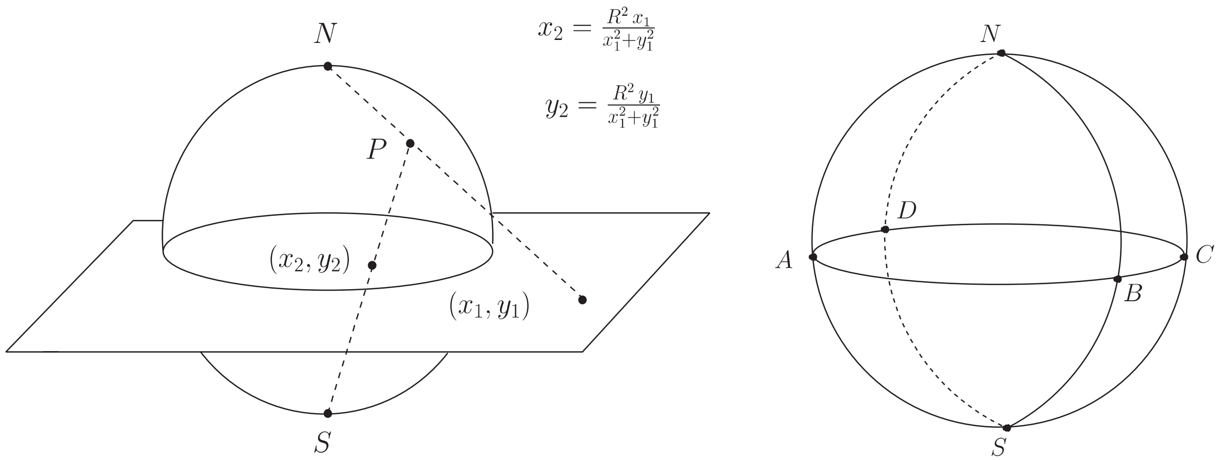

Figure 1: Coordinate charts and triangulation (with 8 triangles) of the sphere of radius *R*.

on the sphere corresponds to a unique point on the plane of the equator: take the straight line *NP* and look at the intersection with the plane (see Figure 1). In other words, the above rule allows us to wrap the equator plane around the sphere in such a way that it will cover all points of the sphere except for the north pole *N*. Similarly, we can do the same thing with the South Pole *S*. In other words, the sphere is made of banding and gluing two planes. Note that the two planes overlap on the sphere everywhere except for *N* and *S*. We can write what was just said in terms of coordinates. Namely the point *P* is obtained by gluing the points with coordinates (x_1, y_1) and (x_2, y_2), which correspond respectively to the intersections of the lines *NP* and *SP* with the equator plane. The coordinates x_2 and y_2 can be expressed in terms of x_1 and y_1 (see Figure 1 for the answer). The precise formula is irrelevant for now. The main point is that the gluing consists of giving a mathematical formula to switch from the coordinates of one linear vector space into another one. The linear vector spaces are called coordinate charts, while the formulas to switch between the coordinates of the charts are called transition functions. The sphere can be constructed from 2 coordinate charts and 1 transition function.

Can we comb a sphere?

It is very difficult to work with coordinates, because the formulas are usually quite cumbersome and the essential properties of the underlying manifold are hard to see. My favorite example is the problem of combing the sphere. Imagine that our sphere has hair, i.e., a piece of hair that comes out of each point on the sphere. Can we make all pieces of hair tangent to the surface of the sphere? The answer is no, and presumably we should be able to prove it using the coordinate charts and the transition functions, but there is a much more elegant approach.

The idea is to think of the tangent bundle of the sphere, i.e., all tangent planes of the sphere. Note that to specify a point on the sphere we need 2 coordinates and to specify a point on the corresponding tangent plane we need yet another 2 coordinates. We obtain a 4-dimensional manifold, so it is one of these things that we can't imagine. Nevertheless, we can clearly visualize an individual tangent plane. If we were able to comb the sphere then by moving the points of the sphere along the corresponding piece of hair we would obtain a surface inside the tangent bundle, which does not intersect the sphere itself. Now one can prove that for every manifold X we can construct a deformation X' of X inside its tangent bundle, such that, X and X' have only isolated points of intersection. Moreover, the number of points of intersection is the Euler characteristic of X. This fact would imply that the sphere has Euler characteristic 0.

The Euler characteristic of a manifold is a bit technical to define, but in the case of surfaces it amounts to triangulating the surface, i.e., choosing several points on the surface and connecting them with curves, so that if we cut the surface along the curves we would obtain (curved) triangles. The Euler characteristic is independent of the choice of a triangulation. It is defined by subtracting the number of curves from the number of points and triangles. For the triangulation depicted on Figure 1, since we have 6 vertices, 12 edges, and 8 triangles, the answer is 6-12+8 = 2, which is the reason why we can't comb the sphere!

Vector bundles and intersection numbers

As pointed out in the above example even if we know explicitly the coordinate charts and the transition functions of some manifold, usually it is very difficult to understand the main properties of the underlying manifold. One of the key ideas in geometry is to build vector bundles by installing a linear vector space, called a fiber, at each point on the manifold. For example, the cylinder and the Möbius strip are line bundles on the circle build by installing a line at each point on the circle (see Figure 2). While for the cylinder the line is installed in the same way, for the Möbius strip, as we move along the circle, the line is rotating clockwise (with respect to the plane of the circle) until it makes a full revolution as we return to the starting point. The vector bundle is also a manifold, but very special one since part of the linear structure of the charts is preserved under gluing. The basic algebraic operations, such as addition and multiplication can be introduced as well, which makes it possible to study the geometry of the underlying manifold by the methods of algebra and to introduce numerical invariants.

Each vector bundle on a given manifold M gives rise to an intersection operation, which produces a new submanifold out of any given submanifold X of M as follows. Let us move the points of M along the fibers of the vector bundle in a continuous fashion, so that we obtain a submanifold Y of the vector bundle, such that, X and Y are transverse to each other. The result of the intersection operation is simply the intersection $X \cap Y$ of X and Y. The transverse property, which will be explained below, is a sufficient condition for $X \cap Y$ to be a submanifold of M contained in X. Starting with any given set of vector bundles, we can successively apply the intersection operations to M: the first intersection operation is applied to M and we obtain a submanifold X_1 of M, the second intersection operation is applied to X_1 and we obtain a submanifold X_2 of X_1, etc. Each time the dimension is decreasing by the rank of the corresponding vector bundle, i.e., the dimension of the fiber. In particular, if the ranks of the vector bundles add up to the dimension of M, then the successive application of the intersection operations yields several isolated points. By counting the number of points we obtain numerical invariants, called intersection numbers.

The continuous deformations of a given submanifold are usually quite many. How to define intersection numbers independent of the choice of the deformations? First, we have to require that our manifolds and vector bundles are orientable. Otherwise, only the parity, i.e., even or odd, of the number of intersection points is well defined. Second, when executing an intersection operation, we are allowed to use only deformations, such that the corresponding intersection is transversal. The main idea behind introducing the notion of

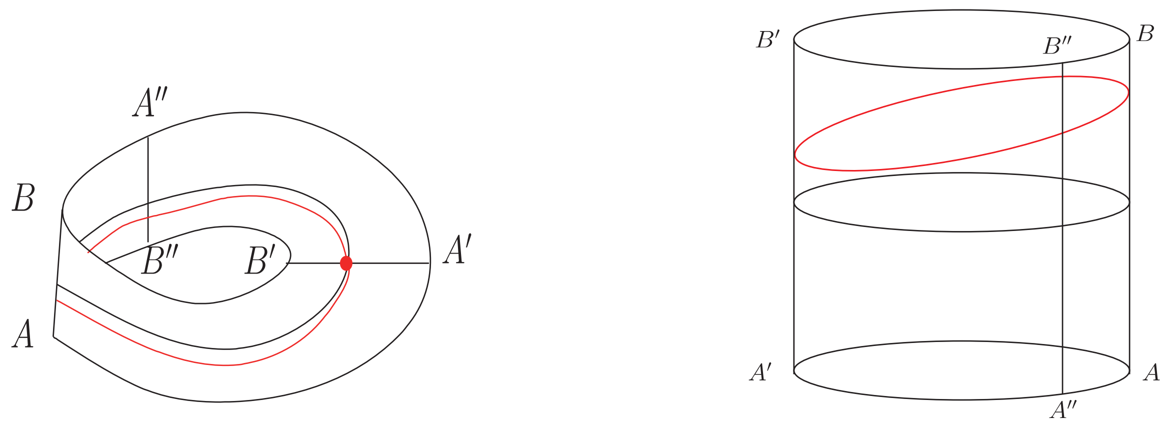

Figure 2: The Möbius strip and the Cylinder as vector bundles. The fibers are the lines *AB*, *A'B'*, *A''B''*, etc. The red curve is a small deformation of the circle along the fibers.

an orientable manifold is the following. If we have two coordinate systems in some linear vector space, then depending on whether we can or can't move continuously one into the other, we can split the set of all coordinate systems into two classes. For example, if we have two coordinate systems in the plane, then we can always move the first coordinate system in such a way that the origins and the 1st axes coincide, while for the 2nd axes there are two possibilities, they either have the same or the opposite directions. We say that the manifold is orientable if the coordinate systems in every two overlapping coordinate charts have the same orientation. Furthermore, two submanifolds of a given manifold intersect transversely if for every point in the intersection we can construct a coordinate system of the coordinate chart of the manifold at that point by using only coordinate axes from the two submanifolds. For example, if two circles in a plane are tangent to each other, then their intersection is not transverse, because the coordinate axes of the two circles at the tangent point have the same direction; so we can't construct from them a coordinate system of the plane. On the other hand, if the circles intersect at two points, then the tangent directions of the two circles at any of the two points of intersection are different; so we can construct a coordinate system, i.e., the intersection is transverse. Finally, we can give the precise definition of an intersection number. If several orientable sub-manifolds intersect transversely in an isolated point, we can compare the orientations of the coordinate system of the manifold and the coordinate system obtained by adjoining the coordinate systems of the submanifolds. Note that the order in which we intersect the submanifolds is important, because this is the order in which we adjoin coordinate systems. If the orientations match then we assign to the intersection point +1, otherwise -1. The intersection number is defined by summing up the numbers associated with all intersection points.

For example, the Möbius strip is not orientable, so it makes sense to ask only about the parity of the intersection number. Moving the circle along the fibers of the Möbius strip (see Figure 2) gives a new circle. The number of points where the two circles intersect is always odd. On the other hand, the cylinder is orientable, so the intersection number is an integer. We can move the circle to a position (see Figure 2) where the two circles do not intersect, so the intersection number must be 0. In other words, the parity of the intersection number can be used to distinguish between the Möbius strip and the cylinder as vector bundles on the circle.

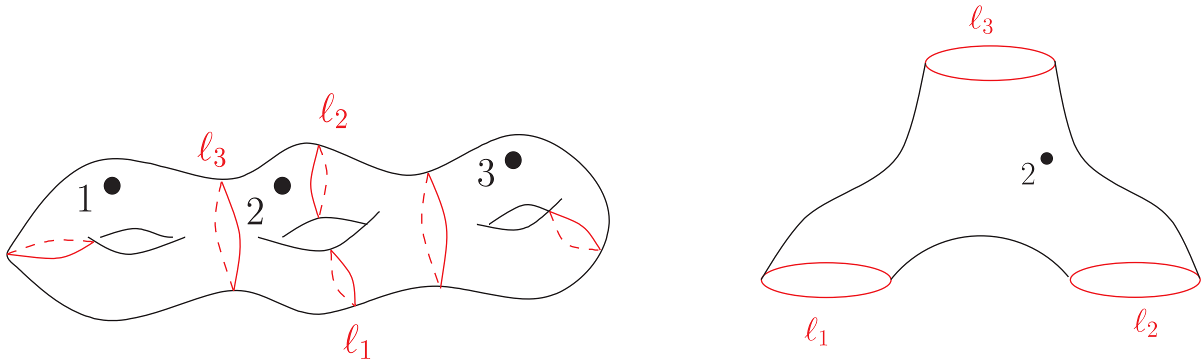

Figure 3: Genus 3 Riemann surface with 3 marked points. Cutting along the red loops gives a pairs of pants decomposition. Note that the number of loops is 3*g*-3 = 6.

The moduli space of Riemann surfaces

As a manifold every surface is uniquely determined from its Euler characteristic. The latter always has the form 2-2*g*, where the number *g* is called the genus of the surface and it coincides with the number of holes. For example, the sphere has Euler characteristic 2 and genus 0, while for the donut the Euler characteristic is 0 (you can take a triangulation and make the same count as we did for the sphere) and genus 1. However, if we are interested in the shape of the surface, then it is important to have a measure for the distance between the points. Usually there is more than 1 way to measure distance and after fixing a measure the surface is called Riemann surface. The basic idea of the moduli space is to give a geometric structure to the entirety of the objects we are trying to classify. The geometry of a single Riemann surface is quite rich and non-trivial, so it is really remarkable that it makes sense to study the set of all Riemann surfaces by using their moduli space. What are the coordinates and what is the dimension of the moduli space of the Riemann surfaces? If you fix two different points on the Riemann surface, then there is a shortest path between them, called geodesic. For example for the sphere the geodesics are precisely the circles whose plane goes through the center of the sphere. The moduli spaces of surfaces of genus *g* = 0, or 1 are much easier to describe. So let us concentrate on the case when *g* is at least 2. One possible approach is to cut the surface along simple closed geodesics in such a way that the surface will decompose into pairs of pants (see Figure 3). By remembering a reference point on each geodesic and measuring its length we can uniquely recover the surface and its metric provided we remember how to glue the different pieces. The moduli space is covered by charts that correspond to the various gluing schemes while the coordinates correspond to the length and the position of the reference point for each geodesic (so 2 parameters for each geodesic). It is not hard to see that the number of simple closed geodesics along which we have to cut the surface is 3*g*-3, so the dimension of the moduli space is 6*g*-6.

It is more convenient, however, to work with slightly more complicated spaces, namely we allow our surfaces to have punctures (also called marked points) and nodes. By forgetting the punctures we can recover the original moduli spaces, while the nodes are necessary in order for the intersection theory to work. Note that fixing a marked point on a surface requires 2 coordinates; so the dimension of the moduli space of genus-*g* Riemann surfaces with *n* marked points is 6*g*-6+2*n*. The moduli space has a natural set of vector bundles corresponding to the marked points. The fiber at a single point, i.e., a

Riemann surface with several marked points, is just the tangent plane of the Riemann surface at the marked point. Using intersection operations similar to the ones discussed earlier we can introduce many intersection numbers that reflect the geometry of the moduli space. What is surprising is that the same intersection numbers can be recovered from the system of differential equations known as the KdV hierarchy.

The KdV integrable hierarchy

The KdV equation is the following partial differential equation $u_t = uu_x + \varepsilon^2 u_{xxx}$, where ε is a parameter whose value could be an arbitrary non-zero number and $u = u(x, t)$ is a function in two variables. I am not going to attempt to describe the history of the KdV equation, but let me just say that it models the motion of a wave in shallow water: t plays the role of time and if we fix t, then the graph of the function $u(x, t)$ with respect to x represents the shape of the wave. The most remarkable feature of the KdV equation is that it can be included into a larger system of equations by allowing additional time variables $t_1 = t$, t_2, t_3, etc. The dependence of the function $u = u(x, t_1, t_2, t_3, \cdots)$ on each additional variable is given by an additional differential equation. We have an entire system of differential equations, which is called the KdV hierarchy. Note that we can't add arbitrary an additional time variable and a corresponding differential equation, because this would usually contradict the previous equations. Starting with the KdV equation there is a unique way to recursively extend the system to include as many time variables as we wish. Equations with such property, i.e., admitting a whole hierarchy of compatible differential equations are called integrable, while the corresponding hierarchy is called integrable hierarchy. Usually, it is very difficult to find integrable differential equations. The equations of the KdV hierarchy become more and more complicated, but every solution depends only on the initial condition, i.e., the shape of the wave when all the time variables are 0. It turns out that if we choose as an initial condition $u(x, 0) = x^3/6$, then the Taylor's series expansion of the solution of the KdV hierarchy determines all intersection numbers on the moduli spaces of Riemann surfaces. The variable t_k corresponds to iterating k times the intersection operation with respect to the line bundles associated with the marked points, while the parameter ε keeps track of the genus of the Riemann surface.

Conclusion

The relation between intersection numbers on the moduli space of Riemann surfaces and the KdV hierarchy was predicted by E. Witten and proved by M. Kontsevich. Nevertheless, it is still very mysterious why does the KdV equation, which we can observe in nature simply by watching the waves in a channel of shallow water, is so important for such a complicated space as the moduli space of Riemann surfaces. Furthermore, we can generalize the moduli space of Riemann surfaces by adding to a given surface the data of all possible ways to fit it (the mathematical word is to map it) in a given manifold. Depending on what manifold we choose we can obtain many other hierarchies of differential equations similar to the KdV hierarchy. These hierarchies are completely new and were never studied before. In fact, in the theory of integrable hierarchies, the construction of an integrable model is quite difficult, so it is very surprising that string theory leads to such a wide class of integrable hierarchies of differential equations. I think that investigating the role of integrability in the geometry of moduli spaces of Riemann surfaces is a very promising direction for the future development of Mathematics.